On the Geometry of a General Three-Camera Headmounted System

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ABSTRACT

A three-dimensional **scene**, such as a proposed building, an imaginary landscape, or an organic molecule, is selected, described in abstract terms, and stored in a computer's memory. A person wears a special **helmet**, in a laboratory whose inner wall is dotted with "landmark" LEDs. The helmet is equipped with a **location system** and a **projection system**. As the wearer moves in the laboratory, changing the helmet's position (and orientation) in a natural manner, the location system allows the computer to keep track of the helmet's position, and the computer sends appropriate information to the projection system, to display the view of the selected scene (suitably scaled) that would be seen by the wearer during this motion.

The geometry of an arbitrarily-arranged, three-camera, headmounted location system for identifying the position and orientation of the helmet, relative to "landmark" pinpoint-LEDs distributed over the inside wall of the laboratory, is described in mathematical terms. A fast Newton-type method for performing the required positional computations, is presented and evaluated.

1. INTRODUCTION

The emotional appeal and practical usefulness of being able to experience what it is like to be somewhere without actually going there (the jungles of New Guinea or Ecuador, the top of Everest, the bottom of the Mindanao Trench, the craters of the Moon, the rooms and hallways of a proposed building, the landscape of a science-fiction adventure, the inside of a watch, a protein molecule, . . .) is evident. The entire film and television industry attests to the idea's popular interest, and the advantages to the trainee-explorer, the anatomist, or the molecular biologist are clearly great. But the viewer of a movie is entirely passive, compelled to follow the motions of the photographer or animator, as the latter chooses the point of view. Some success in allowing the viewer to select and move his or her point of view has been achieved by projecting three-dimensional holographic images; but now the scene to be displayed is very much restricted in space.

The subject of the present work is an alternative approach, in which the viewer walks or rides around a laboratory, as a picture of the selected *scene* is projected for him or her, related to the position and orientation of his or her head, as if the laboratory were magically transformed into the abstract scene. The scene can easily be *scaled*, so that several meters of movement can represent either many thousands of light-years of intergalactic space or a tiny fraction of a micrometer along the surface of a virus molecule. To achieve this effect, the viewer wears a *helmet*, on which are mounted a *location system* and a *projection system*. The inner walls of the laboratory are dotted with "landmark" infra-red LEDs, whose positions are precisely known, and the location system detects the identities, and the positions relative to the helmet, of at least three of these LEDs, so as to determine the wearer's position and orientation in the laboratory (and therefore in the selected scene). Practical considerations, connected with the "engineering balance" between directional sensitivity and breadth of vision, dictate that the location system should consist of *three cameras*, rather than *one*. The LEDs are "lit" briefly, one at a time, in a cyclic manner; so that the location system identifies them by the time at which they are seen.

We are not concerned, here, with the detailed specifications of the equipment, or with the working of the projection system. The focus of the present work is to obtain and analyze a method whereby the angular readings of direction, relative to the helmet, obtained by

the cameras, are combined with the known positions of the LEDs, to yield the position and orientation of the head-mounted system in the laboratory. This work is similar in spirit to that of E. Church (1945, 1948); but the latter was limited to the considerably simpler case of a single camera, not practically useful here, but appropriate to work in aerial photography, for example.

The general problem considered here leads to quite complicated equations, simplified only by their considerable symmetries. To make them manageable, vector transformations which may be unfamiliar to the reader are needed. These and various other mathematical details are collected in the Appendix, for the reader's convenience.

To fix the practical considerations, we note that some 300 readings per second can be made and recorded by the cameras in the computer (this can be increased as much as fivefold). Angular deflections of as much as, say 60° in $\frac{1}{5}$ second, are possible; so that, for successive readings, the computer will have to cope with angular deflections of up to 1° of arc (translational movement is likely to be less abrupt).

2. THE GENERAL THREE-CAMERA SYSTEM

We consider the general geometry shown in Figure 1. Let O be the origin of coordinates. The origin, S, of the headmounted camera system is at (column) vector position $\boldsymbol{s}$ from O. The optical centers of the three camera lenses are at U, V, and W, at vector positions $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$, respectively, from S. Since the headmounted system is rigidly configured, it is clear that, in any position,

$$\begin{bmatrix} \boldsymbol{u}^{\mathrm{T}} \\ \boldsymbol{v}^{\mathrm{T}} \\ \boldsymbol{w}^{\mathrm{T}} \end{bmatrix} [\boldsymbol{u} \;\; \boldsymbol{v} \;\; \boldsymbol{w}] = \begin{bmatrix} \boldsymbol{u}^2 & \boldsymbol{u}.\boldsymbol{v} & \boldsymbol{u}.\boldsymbol{w} \\ \boldsymbol{v}.\boldsymbol{u} & \boldsymbol{v}^2 & \boldsymbol{v}.\boldsymbol{w} \\ \boldsymbol{w}.\boldsymbol{u} & \boldsymbol{w}.\boldsymbol{v} & \boldsymbol{w}^2 \end{bmatrix} = \begin{bmatrix} a & f & e \\ f & b & d \\ e & d & c \end{bmatrix}, \quad (2.1)$$

where [see (A1) - (A8) in the Appendix to this paper] (i) $\boldsymbol{x}^{\mathrm{T}}$ is the *transposed* (row) vector with the same components as the column vector $\boldsymbol{x}$, (ii) $\boldsymbol{x} \,.\, \boldsymbol{y}$ denotes the *scalar product* of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ (in matrix notation, $\boldsymbol{x} \,.\, \boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$), (iii) $\boldsymbol{x}^2 = \boldsymbol{x} \,.\, \boldsymbol{x}$. The parameters a, b, c, d,

e, and f are given by the geometry of the headmounted system (with a, b, and c positive).. In other words, in terms of a "world coordinate frame" of reference $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, say, if

$$\left.\begin{aligned} \boldsymbol{u} &= u_1\,\boldsymbol{i} + u_2\boldsymbol{j} + u_3\,\boldsymbol{k} \\ \boldsymbol{v} &= v_1\,\boldsymbol{i} + v_2\boldsymbol{j} + v_3\,\boldsymbol{k} \\ \boldsymbol{w} &= w_1\,\boldsymbol{i} + w_2\boldsymbol{j} + w_3\,\boldsymbol{k} \end{aligned}\right\}, \tag{2.2}$$

i.e.,

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}; \tag{2.3}$$

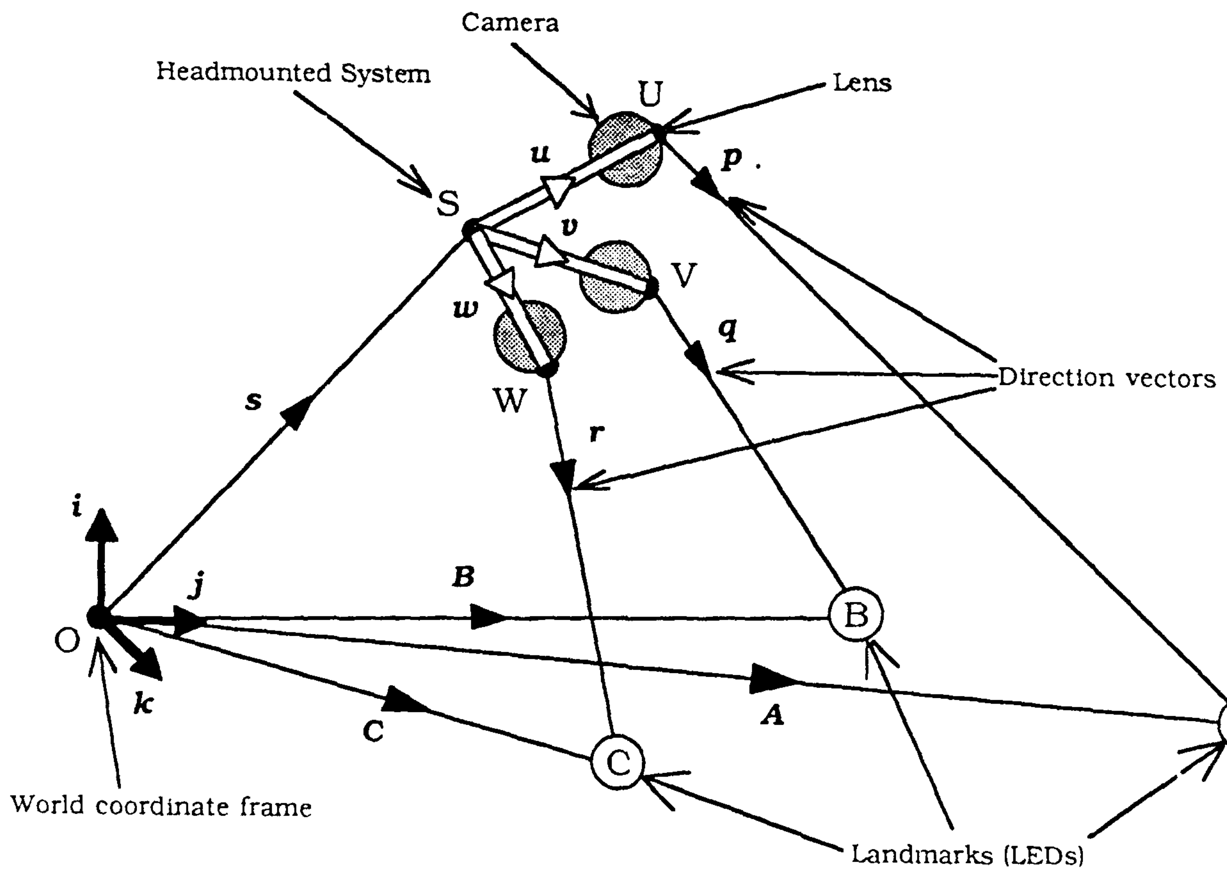

Figure 1.

then

$$\left.\begin{aligned} \boldsymbol{u}^2 &= u_1{}^2 + u_2{}^2 + u_3{}^2 = a \\ \boldsymbol{v}^2 &= v_1{}^2 + v_2{}^2 + v_3{}^2 = b \\ \boldsymbol{w}^2 &= w_1{}^2 + w_2{}^2 + w_3{}^2 = c \end{aligned}\right\}, \tag{2.4a}$$

$$\left.\begin{aligned} \boldsymbol{v}\,.\,\boldsymbol{w} = \boldsymbol{w}\,.\,\boldsymbol{v} &= v_1w_1 + v_2w_2 + v_3w_3 = d \\ \boldsymbol{w}\,.\,\boldsymbol{u} = \boldsymbol{u}\,.\,\boldsymbol{w} &= w_1u_1 + w_2u_2 + w_3u_3 = e \\ \boldsymbol{u}\,.\,\boldsymbol{v} = \boldsymbol{v}\,.\,\boldsymbol{u} &= u_1v_1 + u_2v_2 + u_3v_3 = f \end{aligned}\right\}. \tag{2.4b}$$

The "landmarks" (LEDs), A, B, and C, are respectively sighted from the cameras at U, V, and W, in the directions of vectors $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{r}$. However, only the *directions* of these vectors are significant (the magnitudes are arbitrary). We note that $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{r}$ will be determined relative to their respective cameras' orientations. Thus, by suitably transforming the raw experimental observations, we may view these (column) vectors as satisfying a relation of the form

$$\left.\begin{aligned} \boldsymbol{p} &= k_{11}\,\boldsymbol{u} + k_{21}\,\boldsymbol{v} + k_{31}\,\boldsymbol{w} \\ \boldsymbol{q} &= k_{12}\,\boldsymbol{u} + k_{22}\,\boldsymbol{v} + k_{32}\,\boldsymbol{w} \\ \boldsymbol{r} &= k_{13}\,\boldsymbol{u} + k_{23}\,\boldsymbol{v} + k_{33}\,\boldsymbol{w} \end{aligned}\right\}, \tag{2.5}$$

where the nine components k_{ij} of the square matrix $\boldsymbol{K}$ (which will be *invertible* so long as the vectors $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ form a *base*) are all known even though $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are not. Equation (2.5) may be written as

$$[\boldsymbol{p} \quad \boldsymbol{q} \quad \boldsymbol{r}] = [\boldsymbol{u} \quad \boldsymbol{v} \quad \boldsymbol{w}]\,\boldsymbol{K}, \tag{2.6}$$

Note that, since $\boldsymbol{K}$ is invertible (with reciprocal $\boldsymbol{K}^{-1} = \boldsymbol{H}$, say), (2.5) or (2.6) implies that

$$[\boldsymbol{u} \quad \boldsymbol{v} \quad \boldsymbol{w}] = [\boldsymbol{p} \quad \boldsymbol{q} \quad \boldsymbol{r}]\,\boldsymbol{H}; \tag{2.7}$$

i.e.,

$$\left.\begin{aligned} \boldsymbol{u} &= h_{11}\,\boldsymbol{p} + h_{21}\,\boldsymbol{q} + h_{31}\,\boldsymbol{r} \\ \boldsymbol{v} &= h_{12}\,\boldsymbol{p} + h_{22}\,\boldsymbol{q} + h_{32}\,\boldsymbol{r} \\ \boldsymbol{w} &= h_{13}\,\boldsymbol{p} + h_{23}\,\boldsymbol{q} + h_{33}\,\boldsymbol{r} \end{aligned}\right\}. \tag{2.8}$$

Now, we are given the actual spatial positions—relative to the origin O—of A, B, and C; namely, **A**, **B**, and **C**, respectively. Thus, we know that multipliers λ, μ, and ν exist, for which

$$\left.\begin{aligned} \boldsymbol{s}+\boldsymbol{u}+\lambda\boldsymbol{p} &= \boldsymbol{A} \\ \boldsymbol{s}+\boldsymbol{v}+\mu\boldsymbol{q} &= \boldsymbol{B} \\ \boldsymbol{s}+\boldsymbol{w}+\nu\boldsymbol{r} &= \boldsymbol{C} \end{aligned}\right\}. \qquad (2.9)$$

If

$$\boldsymbol{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}; \qquad (2.10)$$

then (2.9) becomes

$$\left.\begin{aligned} s_1 + u_1 + \lambda p_1 &= A_1 \\ s_2 + u_2 + \lambda p_2 &= A_2 \\ s_3 + u_3 + \lambda p_3 &= A_3 \end{aligned}\right\}, \qquad (2.11a)$$

$$\left.\begin{aligned} s_1 + v_1 + \mu q_1 &= B_1 \\ s_2 + v_2 + \mu q_2 &= B_2 \\ s_3 + v_3 + \mu q_3 &= B_3 \end{aligned}\right\}, \qquad (2.11b)$$

$$\left.\begin{aligned} s_1 + w_1 + \nu r_1 &= C_1 \\ s_2 + w_2 + \nu r_2 &= C_2 \\ s_3 + w_3 + \nu r_3 &= C_3 \end{aligned}\right\}. \qquad (2.11c)$$

There are thus altogether 15 scalar unknowns; namely, λ, μ, ν, and the components of the four 3-vectors **s**, **u**, **v**, and **w**. To determine these, we have six independent equations in (2.4) and nine equations in (2.11).

We can now proceed by eliminating, first, λ, μ, and ν, and then s_1, s_2, and s_3 from the equations in (2.11), to yield three equations;

but a purely vectorial approach gives us the same results more directly. First, we vector-multiply the equations in (2.9) by $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{r}$, respectively, to eliminate from these equations the unneeded parameters λ, μ, and ν:

$$\left.\begin{aligned} \boldsymbol{s} \wedge \boldsymbol{p} &= (\boldsymbol{A} - \boldsymbol{u}) \wedge \boldsymbol{p} \\ \boldsymbol{s} \wedge \boldsymbol{q} &= (\boldsymbol{B} - \boldsymbol{v}) \wedge \boldsymbol{q} \\ \boldsymbol{s} \wedge \boldsymbol{r} &= (\boldsymbol{C} - \boldsymbol{w}) \wedge \boldsymbol{r} \end{aligned}\right\}, \qquad (2.12)$$

where [see (A9) - (A14) in the Appendix] $\boldsymbol{x} \wedge \boldsymbol{y}$ denotes the anti-commutative *vector product* of $\boldsymbol{x}$ and $\boldsymbol{y}$. Then we scalar-multiply the equations in (2.12) by $\boldsymbol{q}$ and $\boldsymbol{r}$, $\boldsymbol{r}$ and $\boldsymbol{p}$, and $\boldsymbol{p}$ and $\boldsymbol{q}$, respectively [using (A15) and (A16) of the Appendix], to eliminate $\boldsymbol{s}$:

$$\left.\begin{aligned} |\,\boldsymbol{s}\ \ \boldsymbol{q}\ \ \boldsymbol{r}\,| &= |\,(\boldsymbol{B} - \boldsymbol{v})\ \ \boldsymbol{q}\ \ \boldsymbol{r}\,| = |\,(\boldsymbol{C} - \boldsymbol{w})\ \ \boldsymbol{q}\ \ \boldsymbol{r}\,| \\ |\,\boldsymbol{s}\ \ \boldsymbol{r}\ \ \boldsymbol{p}\,| &= |\,(\boldsymbol{C} - \boldsymbol{w})\ \ \boldsymbol{r}\ \ \boldsymbol{p}\,| = |\,(\boldsymbol{A} - \boldsymbol{u})\ \ \boldsymbol{r}\ \ \boldsymbol{p}\,| \\ |\,\boldsymbol{s}\ \ \boldsymbol{p}\ \ \boldsymbol{q}\,| &= |\,(\boldsymbol{A} - \boldsymbol{u})\ \ \boldsymbol{p}\ \ \boldsymbol{q}\,| = |\,(\boldsymbol{B} - \boldsymbol{v})\ \ \boldsymbol{p}\ \ \boldsymbol{q}\,| \end{aligned}\right\}. \qquad (2.13)$$

Using (A17)—Fact 4—of the Appendix, and (2.13), we see that

$$\boldsymbol{s} = \frac{|\,(\boldsymbol{B} - \boldsymbol{v})\ \ \boldsymbol{q}\ \ \boldsymbol{r}\,|}{|\,\boldsymbol{p}\ \ \boldsymbol{q}\ \ \boldsymbol{r}\,|}\boldsymbol{p} + \frac{|\,(\boldsymbol{C} - \boldsymbol{w})\ \ \boldsymbol{r}\ \ \boldsymbol{p}\,|}{|\,\boldsymbol{p}\ \ \boldsymbol{q}\ \ \boldsymbol{r}\,|}\boldsymbol{q} + \frac{|\,(\boldsymbol{A} - \boldsymbol{u})\ \ \boldsymbol{p}\ \ \boldsymbol{q}\,|}{|\,\boldsymbol{p}\ \ \boldsymbol{q}\ \ \boldsymbol{r}\,|}\boldsymbol{r}, \qquad (2.14)$$

which will enable us to recover $\boldsymbol{s}$ when we know $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{r}$.

Now, by (2.5) with (A11), we see that

$$\begin{aligned} \boldsymbol{q} \wedge \boldsymbol{r} &= (k_{12}\,\boldsymbol{u} + k_{22}\,\boldsymbol{v} + k_{32}\,\boldsymbol{w}) \wedge (k_{13}\,\boldsymbol{u} + k_{23}\,\boldsymbol{v} + k_{33}\,\boldsymbol{w}) \\ &= (k_{22}k_{33} - k_{23}k_{32})\,\boldsymbol{v} \wedge \boldsymbol{w} + (k_{32}k_{13} - k_{33}k_{12})\,\boldsymbol{w} \wedge \boldsymbol{u} \\ &\quad + (k_{12}k_{23} - k_{13}k_{22})\,\boldsymbol{u} \wedge \boldsymbol{v}. \end{aligned} \qquad (2.15)$$

Hence and by analogy, using (A24) and (A26), we see that

$$\left.\begin{aligned} \boldsymbol{q}\wedge\boldsymbol{r} &= |\boldsymbol{K}|(h_{11}\,\boldsymbol{v}\wedge\boldsymbol{w}+h_{12}\,\boldsymbol{w}\wedge\boldsymbol{u}+h_{13}\,\boldsymbol{u}\wedge\boldsymbol{v}) \\ \boldsymbol{r}\wedge\boldsymbol{p} &= |\boldsymbol{K}|(h_{21}\,\boldsymbol{v}\wedge\boldsymbol{w}+h_{22}\,\boldsymbol{w}\wedge\boldsymbol{u}+h_{23}\,\boldsymbol{u}\wedge\boldsymbol{v}) \\ \boldsymbol{p}\wedge\boldsymbol{q} &= |\boldsymbol{K}|(h_{31}\,\boldsymbol{v}\wedge\boldsymbol{w}+h_{32}\,\boldsymbol{w}\wedge\boldsymbol{u}+h_{33}\,\boldsymbol{u}\wedge\boldsymbol{v}) \end{aligned}\right\}. \tag{2.16}$$

Since, by (A15), $$|\,\boldsymbol{x}\;\;\boldsymbol{y}\;\;\boldsymbol{z}\,| = \boldsymbol{x}\,.\,(\boldsymbol{y}\wedge\boldsymbol{z}); \tag{2.17}$$

it follows from (2.16) that

$$\left.\begin{aligned} |\,\boldsymbol{x}\;\;\boldsymbol{q}\;\;\boldsymbol{r}\,| &= |\boldsymbol{K}|\;\boldsymbol{x}\,.\,(h_{11}\,\boldsymbol{v}\wedge\boldsymbol{w}+h_{12}\,\boldsymbol{w}\wedge\boldsymbol{u}+h_{13}\,\boldsymbol{u}\wedge\boldsymbol{v}) \\ |\,\boldsymbol{y}\;\;\boldsymbol{r}\;\;\boldsymbol{p}\,| &= |\boldsymbol{K}|\;\boldsymbol{y}\,.\,(h_{21}\,\boldsymbol{v}\wedge\boldsymbol{w}+h_{22}\,\boldsymbol{w}\wedge\boldsymbol{u}+h_{23}\,\boldsymbol{u}\wedge\boldsymbol{v}) \\ |\,\boldsymbol{z}\;\;\boldsymbol{p}\;\;\boldsymbol{q}\,| &= |\boldsymbol{K}|\;\boldsymbol{z}\,.\,(h_{31}\,\boldsymbol{v}\wedge\boldsymbol{w}+h_{32}\,\boldsymbol{w}\wedge\boldsymbol{u}+h_{33}\,\boldsymbol{u}\wedge\boldsymbol{v}) \end{aligned}\right\}. \tag{2.18}$$

The equations in (2.13) *not* involving $\boldsymbol{s}$ are

$$\left.\begin{aligned} |\,(\boldsymbol{B}-\boldsymbol{C}+\boldsymbol{w}-\boldsymbol{v})\;\;\boldsymbol{q}\;\;\boldsymbol{r}\,| &= 0 \\ |\,(\boldsymbol{C}-\boldsymbol{A}+\boldsymbol{u}-\boldsymbol{w})\;\;\boldsymbol{r}\;\;\boldsymbol{p}\,| &= 0 \\ |\,(\boldsymbol{A}-\boldsymbol{B}+\boldsymbol{v}-\boldsymbol{u})\;\;\boldsymbol{p}\;\;\boldsymbol{q}\,| &= 0 \end{aligned}\right\}; \tag{2.19}$$

and, with (2.18), these yield

$$\left.\begin{aligned} |\boldsymbol{K}|\;(\boldsymbol{B}-\boldsymbol{C}+\boldsymbol{w}-\boldsymbol{v})\,.\,(h_{11}\,\boldsymbol{v}\wedge\boldsymbol{w}+h_{12}\,\boldsymbol{w}\wedge\boldsymbol{u}+h_{13}\,\boldsymbol{u}\wedge\boldsymbol{v}) &= 0 \\ |\boldsymbol{K}|\;(\boldsymbol{C}-\boldsymbol{A}+\boldsymbol{u}-\boldsymbol{w})\,.\,(h_{21}\,\boldsymbol{v}\wedge\boldsymbol{w}+h_{22}\,\boldsymbol{w}\wedge\boldsymbol{u}+h_{23}\,\boldsymbol{u}\wedge\boldsymbol{v}) &= 0 \\ |\boldsymbol{K}|\;(\boldsymbol{A}-\boldsymbol{B}+\boldsymbol{v}-\boldsymbol{u})\,.\,(h_{31}\,\boldsymbol{v}\wedge\boldsymbol{w}+h_{32}\,\boldsymbol{w}\wedge\boldsymbol{u}+h_{33}\,\boldsymbol{u}\wedge\boldsymbol{v}) &= 0 \end{aligned}\right\}. \tag{2.20}$$

If we write $$|\,\boldsymbol{u}\;\;\boldsymbol{v}\;\;\boldsymbol{w}\,| = \Delta, \tag{2.21}$$

we see that (since the determinant of a product of matrices equals the product of the determinants of the factor matrices, and since determinants are not altered by transposition), by (2.1), Δ will satisfy the equation

$$\Delta^2 = |\,\boldsymbol{u} \quad \boldsymbol{v} \quad \boldsymbol{w}\,|^2 = \begin{vmatrix} a & d & f \\ d & b & e \\ f & e & c \end{vmatrix}; \tag{2.22}$$

and, further, by (2.6),

$$|\,\boldsymbol{p} \quad \boldsymbol{q} \quad \boldsymbol{r}\,| = \Delta|\boldsymbol{K}|. \tag{2.23}$$

Let us also write

$$\left.\begin{array}{lll} D_1 = |\,(\boldsymbol{B}-\boldsymbol{C}) \;\; \boldsymbol{v} \;\; \boldsymbol{w}\,|, & D_2 = |\,(\boldsymbol{B}-\boldsymbol{C}) \;\; \boldsymbol{w} \;\; \boldsymbol{u}\,|, & D_3 = |\,(\boldsymbol{B}-\boldsymbol{C}) \;\; \boldsymbol{u} \;\; \boldsymbol{v}\,| \\ E_1 = |\,(\boldsymbol{C}-\boldsymbol{A}) \;\; \boldsymbol{v} \;\; \boldsymbol{w}\,|, & E_2 = |\,(\boldsymbol{C}-\boldsymbol{A}) \;\; \boldsymbol{w} \;\; \boldsymbol{u}\,|, & E_3 = |\,(\boldsymbol{C}-\boldsymbol{A}) \;\; \boldsymbol{u} \;\; \boldsymbol{v}\,| \\ F_1 = |\,(\boldsymbol{A}-\boldsymbol{B}) \;\; \boldsymbol{v} \;\; \boldsymbol{w}\,|, & F_2 = |\,(\boldsymbol{A}-\boldsymbol{B}) \;\; \boldsymbol{w} \;\; \boldsymbol{u}\,|, & F_3 = |\,(\boldsymbol{A}-\boldsymbol{B}) \;\; \boldsymbol{u} \;\; \boldsymbol{v}\,| \end{array}\right\}. \tag{2.24}$$

Then (2.20) simplifies to

$$\left.\begin{array}{rl} h_{11}D_1 + h_{12}D_2 + h_{13}D_3 &= \Delta(h_{12} - h_{13}) \\ h_{21}E_1 + h_{22}E_2 + h_{23}E_3 &= \Delta(h_{23} - h_{21}) \\ h_{31}F_1 + h_{32}F_2 + h_{33}F_3 &= \Delta(h_{31} - h_{32}) \end{array}\right\}. \tag{2.25}$$

3. NEWTON'S METHOD

We see, by (2.24), that equations (2.4) and (2.25) are quadratic in the components of $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$, all the coefficients being known (or, at least, easily computed). It is easiest to solve these equations by *Newton's method*. We begin the computation with an initial approximation, $\{\boldsymbol{u}^{(0)}, \boldsymbol{v}^{(0)}, \boldsymbol{w}^{(0)}\}$, which can, for instance, be the solution obtained (by Newton's method) for the most recent set of observations, as the wearer of the headmounted system moves about

the laboratory. (For the very first set of observations, some kind of initial guess must be used.) We now suppose that we have an approximate solution, $\{\boldsymbol{u}^{(m)}, \boldsymbol{v}^{(m)}, \boldsymbol{w}^{(m)}\}$, after m iterations. We define increments $\{\delta\boldsymbol{u}^{(m)}, \delta\boldsymbol{v}^{(m)}, \delta\boldsymbol{w}^{(m)}\}$ by

$$\left.\begin{aligned} \delta\boldsymbol{u}^{(m)} &= \boldsymbol{u}^{(m+1)} - \boldsymbol{u}^{(m)} \\ \delta\boldsymbol{v}^{(m)} &= \boldsymbol{v}^{(m+1)} - \boldsymbol{v}^{(m)} \\ \delta\boldsymbol{w}^{(m)} &= \boldsymbol{w}^{(m+1)} - \boldsymbol{w}^{(m)} \end{aligned}\right\}. \tag{3.1a}$$

Since the superscripts "(m)" and "$(m+1)$" will appear very frequently, from now on, we shall replace them by primes (′ and ″ respectively). Then (3.1a) will take the form

$$\left.\begin{aligned} \delta\boldsymbol{u}' &= \boldsymbol{u}'' - \boldsymbol{u}' \\ \delta\boldsymbol{v}' &= \boldsymbol{v}'' - \boldsymbol{v}' \\ \delta\boldsymbol{w}' &= \boldsymbol{w}'' - \boldsymbol{w}' \end{aligned}\right\}. \tag{3.1b}$$

Newton's method consists of *linearizing* the problem, and solving for the increments which will reduce the discrepancies in the equations to zero. The linearized difference equations arising from (2.4) are

$$\left.\begin{aligned} \boldsymbol{u}' \,.\, \boldsymbol{u}' + 2\boldsymbol{u}' \,.\, \delta\boldsymbol{u}' &= a \\ \boldsymbol{v}' \,.\, \boldsymbol{v}' + 2\boldsymbol{v}' \,.\, \delta\boldsymbol{v}' &= b \\ \boldsymbol{w}' \,.\, \boldsymbol{w}' + 2\boldsymbol{w}' \,.\, \delta\boldsymbol{w}' &= c \end{aligned}\right\}, \tag{3.2a}$$

$$\left.\begin{aligned} \boldsymbol{v}' \,.\, \boldsymbol{w}' + \delta\boldsymbol{v}' \,.\, \boldsymbol{w}' + \boldsymbol{v}' \,.\, \delta\boldsymbol{w}' &= d \\ \boldsymbol{w}' \,.\, \boldsymbol{u}' + \delta\boldsymbol{w}' \,.\, \boldsymbol{u}' + \boldsymbol{w}' \,.\, \delta\boldsymbol{u}' &= e \\ \boldsymbol{u}' \,.\, \boldsymbol{v}' + \delta\boldsymbol{u}' \,.\, \boldsymbol{v}' + \boldsymbol{u}' \,.\, \delta\boldsymbol{v}' &= f \end{aligned}\right\}; \tag{3.2b}$$

and, by (2.24) and (2.25), we have

$$h_{11}\left[\left|(\boldsymbol{B}-\boldsymbol{C})\ \boldsymbol{v}'\ \boldsymbol{w}'\right| + \left|(\boldsymbol{B}-\boldsymbol{C})\ \delta\boldsymbol{v}'\ \boldsymbol{w}'\right| + \left|(\boldsymbol{B}-\boldsymbol{C})\ \boldsymbol{v}'\ \delta\boldsymbol{w}'\right|\right]$$
$$+ h_{12}\left[\left|(\boldsymbol{B}-\boldsymbol{C})\ \boldsymbol{w}'\ \boldsymbol{u}'\right| + \left|(\boldsymbol{B}-\boldsymbol{C})\ \delta\boldsymbol{w}'\ \boldsymbol{u}'\right| + \left|(\boldsymbol{B}-\boldsymbol{C})\ \boldsymbol{w}'\ \delta\boldsymbol{u}'\right|\right]$$
$$+ h_{13}\left[\left|(\boldsymbol{B}-\boldsymbol{C})\ \boldsymbol{u}'\ \boldsymbol{v}'\right| + \left|(\boldsymbol{B}-\boldsymbol{C})\ \delta\boldsymbol{u}'\ \boldsymbol{v}'\right| + \left|(\boldsymbol{B}-\boldsymbol{C})\ \boldsymbol{u}'\ \delta\boldsymbol{v}'\right|\right]$$
$$= \Delta(h_{12} - h_{13}). \tag{3.2c}$$

$$h_{21}\left[\left|(\boldsymbol{C}-\boldsymbol{A})\ \boldsymbol{v}'\ \boldsymbol{w}'\right| + \left|(\boldsymbol{C}-\boldsymbol{A})\ \delta\boldsymbol{v}'\ \boldsymbol{w}'\right| + \left|(\boldsymbol{C}-\boldsymbol{A})\ \boldsymbol{v}'\ \delta\boldsymbol{w}'\right|\right]$$
$$+ h_{22}\left[\left|(\boldsymbol{C}-\boldsymbol{A})\ \boldsymbol{w}'\ \boldsymbol{u}'\right| + \left|(\boldsymbol{C}-\boldsymbol{A})\ \delta\boldsymbol{w}'\ \boldsymbol{u}'\right| + \left|(\boldsymbol{C}-\boldsymbol{A})\ \boldsymbol{w}'\ \delta\boldsymbol{u}'\right|\right]$$
$$+ h_{23}\left[\left|(\boldsymbol{C}-\boldsymbol{A})\ \boldsymbol{u}'\ \boldsymbol{v}'\right| + \left|(\boldsymbol{C}-\boldsymbol{A})\ \delta\boldsymbol{u}'\ \boldsymbol{v}'\right| + \left|(\boldsymbol{C}-\boldsymbol{A})\ \boldsymbol{u}'\ \delta\boldsymbol{v}'\right|\right]$$
$$= \Delta(h_{23} - h_{21}). \tag{3.2d}$$

$$h_{31}\left[\left|(\boldsymbol{A}-\boldsymbol{B})\ \boldsymbol{v}'\ \boldsymbol{w}'\right| + \left|(\boldsymbol{A}-\boldsymbol{B})\ \delta\boldsymbol{v}'\ \boldsymbol{w}'\right| + \left|(\boldsymbol{A}-\boldsymbol{B})\ \boldsymbol{v}'\ \delta\boldsymbol{w}'\right|\right]$$
$$+ h_{32}\left[\left|(\boldsymbol{A}-\boldsymbol{B})\ \boldsymbol{w}'\ \boldsymbol{u}'\right| + \left|(\boldsymbol{A}-\boldsymbol{B})\ \delta\boldsymbol{w}'\ \boldsymbol{u}'\right| + \left|(\boldsymbol{A}-\boldsymbol{B})\ \boldsymbol{w}'\ \delta\boldsymbol{u}'\right|\right]$$
$$+ h_{33}\left[\left|(\boldsymbol{A}-\boldsymbol{B})\ \boldsymbol{u}'\ \boldsymbol{v}'\right| + \left|(\boldsymbol{A}-\boldsymbol{B})\ \delta\boldsymbol{u}'\ \boldsymbol{v}'\right| + \left|(\boldsymbol{A}-\boldsymbol{B})\ \boldsymbol{u}'\ \delta\boldsymbol{v}'\right|\right]$$
$$= \Delta(h_{31} - h_{32}). \tag{3.2e}$$

While these equations are quite long, they nevertheless reduce to nine simultaneous linear equations for the nine unknown increments, with coefficients which are relatively simple linear combinations of the components of the previous iterate. We may further simplify the equations as follows. First, consider equations (3.2a) and (3.2b). Let us put

$$\left.\begin{array}{lll} a^* = \dfrac{a - \boldsymbol{u}' \cdot \boldsymbol{u}'}{2\Delta}, & b^* = \dfrac{b - \boldsymbol{v}' \cdot \boldsymbol{v}'}{2\Delta}, & c^* = \dfrac{c - \boldsymbol{w}' \cdot \boldsymbol{w}'}{2\Delta} \\ d^* = \dfrac{d - \boldsymbol{v}' \cdot \boldsymbol{w}'}{\Delta}, & e^* = \dfrac{e - \boldsymbol{w}' \cdot \boldsymbol{u}'}{\Delta}, & f^* = \dfrac{f - \boldsymbol{u}' \cdot \boldsymbol{v}'}{\Delta} \end{array}\right\}, \tag{3.3}$$

where the asterisk (like the primes) denotes a dependence on m; and write

$$\left.\begin{aligned}\delta u' &= \delta'_{11}\, v' \wedge w' + \delta'_{21}\, w' \wedge u' + \delta'_{31}\, u' \wedge v' \\ \delta v' &= \delta'_{12}\, v' \wedge w' + \delta'_{22}\, w' \wedge u' + \delta'_{32}\, u' \wedge v' \\ \delta w' &= \delta'_{13}\, v' \wedge w' + \delta'_{23}\, w' \wedge u' + \delta'_{33}\, u' \wedge v'\end{aligned}\right\}. \tag{3.4}$$

Then, by Fact 2 of the Appendix and (2.21), on suitably scalar-multiplying (3.4), we see that

$$\delta'_{11} = a^*, \quad \delta'_{22} = b^*, \quad \delta'_{33} = c^*, \tag{3.5a}$$

and

$$\left.\begin{aligned}\delta'_{23} + \delta'_{32} &= d^* \\ \delta'_{31} + \delta'_{13} &= e^* \\ \delta'_{12} + \delta'_{21} &= f^*\end{aligned}\right\}; \tag{3.5b}$$

which reduces (3.4) to

$$\left.\begin{aligned}\delta u' &= a^*\, v' \wedge w' + \xi\, w' \wedge u' + (e^* - \zeta)\, u' \wedge v' \\ \delta v' &= (f^* - \xi)\, v' \wedge w' + b^*\, w' \wedge u' + \eta\, u' \wedge v' \\ \delta w' &= \zeta\, v' \wedge w' + (d^* - \eta)\, w' \wedge u' + c^*\, u' \wedge v'\end{aligned}\right\}, \tag{3.6}$$

where $\xi = \delta'_{21}, \quad \eta = \delta'_{32}, \quad \zeta = \delta'_{13}\,.$ (3.7)

We are now down to three equations, (3.2c) - (3.2e), in the remaining three unknowns, ξ, η, and ζ. From (3.6), we get, by (A19), that

$$\begin{aligned}\delta v' \wedge w' &= (f^* - \xi)\,(v' \wedge w') \wedge w' + b^*\,(w' \wedge u') \wedge w' + \eta\,(u' \wedge v') \wedge w' \\ &= [b^*\,(w'^2) - \eta\,(v' \cdot w')]\, u' + [\eta\,(w' \cdot u') - (f^* - \xi)\,(w'^2)]\, v' \\ &\quad + [(f^* - \xi)\,(v' \cdot w') - b^*\,(w' \cdot u')]\, w',\end{aligned} \tag{3.8a}$$

$$\begin{aligned}v' \wedge \delta w' &= \zeta\, v' \wedge (v' \wedge w') + (d^* - \eta)\, v' \wedge (w' \wedge u') + c^*\, v' \wedge (u' \wedge v') \\ &= [c^*\,(v'^2) - (d^* - \eta)\,(v' \cdot w')]\, u' + [\zeta\,(v' \cdot w') - c^*\,(u' \cdot v')]\, v' \\ &\quad + [(d^* - \eta)\,(u' \cdot v') - \zeta\,(v'^2)]\, w'.\end{aligned} \tag{3.8b}$$

$$\begin{aligned}\delta \boldsymbol{w}' \wedge \boldsymbol{u}' &= \zeta\,(\boldsymbol{v}' \wedge \boldsymbol{w}') \wedge \boldsymbol{u}' + (d^* - \eta)\,(\boldsymbol{w}' \wedge \boldsymbol{u}') \wedge \boldsymbol{u}' + c^*\,(\boldsymbol{u}' \wedge \boldsymbol{v}') \wedge \boldsymbol{u}' \\ &= [(d^* - \eta)\,(\boldsymbol{w}' \cdot \boldsymbol{u}') - c^*\,(\boldsymbol{u}' \cdot \boldsymbol{v}')]\,\boldsymbol{u}' + [c^*\,(\boldsymbol{u}'^2) - \zeta\,(\boldsymbol{w}' \cdot \boldsymbol{u}')]\,\boldsymbol{v}' \\ &\quad + [\zeta\,(\boldsymbol{u}' \cdot \boldsymbol{v}') - (d^* - \eta)\,(\boldsymbol{u}'^2)]\,\boldsymbol{w}', \end{aligned} \tag{3.8c}$$

$$\begin{aligned}\boldsymbol{w}' \wedge \delta \boldsymbol{u}' &= a^*\,\boldsymbol{w}' \wedge (\boldsymbol{v}' \wedge \boldsymbol{w}') + \xi\,\boldsymbol{w}' \wedge (\boldsymbol{w}' \wedge \boldsymbol{u}') + (e^* - \zeta)\,\boldsymbol{w}' \wedge (\boldsymbol{u}' \wedge \boldsymbol{v}') \\ &= [(e^* - \zeta)\,(\boldsymbol{v}' \cdot \boldsymbol{w}') - \xi\,(\boldsymbol{w}'^2)]\,\boldsymbol{u}' + [a^*\,(\boldsymbol{w}'^2) - (e^* - \zeta)\,(\boldsymbol{w}' \cdot \boldsymbol{u}')]\,\boldsymbol{v}' \\ &\quad + [\xi\,(\boldsymbol{w}' \cdot \boldsymbol{u}') - a^*\,(\boldsymbol{v}' \cdot \boldsymbol{w}')]\,\boldsymbol{w}', \end{aligned} \tag{3.8d}$$

$$\begin{aligned}\delta \boldsymbol{u}' \wedge \boldsymbol{v}' &= a^*\,(\boldsymbol{v}' \wedge \boldsymbol{w}') \wedge \boldsymbol{v}' + \xi\,(\boldsymbol{w}' \wedge \boldsymbol{u}') \wedge \boldsymbol{v}' + (e^* - \zeta)\,(\boldsymbol{u}' \wedge \boldsymbol{v}') \wedge \boldsymbol{v}' \\ &= [\xi\,(\boldsymbol{v}' \cdot \boldsymbol{w}') - (e^* - \zeta)\,(\boldsymbol{v}'^2)]\,\boldsymbol{u}' + [(e^* - \zeta)\,(\boldsymbol{u}' \cdot \boldsymbol{v}') - a^*\,(\boldsymbol{v}' \cdot \boldsymbol{w}')]\,\boldsymbol{v}' \\ &\quad + [a^*\,(\boldsymbol{v}'^2) - \xi\,(\boldsymbol{u}' \cdot \boldsymbol{v}')]\,\boldsymbol{w}', \end{aligned} \tag{3.8e}$$

$$\begin{aligned}\boldsymbol{u}' \wedge \delta \boldsymbol{v}' &= (f^* - \xi)\,\boldsymbol{u}' \wedge (\boldsymbol{v}' \wedge \boldsymbol{w}') + b^*\,\boldsymbol{u}' \wedge (\boldsymbol{w}' \wedge \boldsymbol{u}') + \eta\,\boldsymbol{u}' \wedge (\boldsymbol{u}' \wedge \boldsymbol{v}') \\ &= [\eta\,(\boldsymbol{u}' \cdot \boldsymbol{v}') - b^*\,(\boldsymbol{w}' \cdot \boldsymbol{u}')]\,\boldsymbol{u}' + [(f^* - \xi)\,(\boldsymbol{w}' \cdot \boldsymbol{u}') - \eta\,(\boldsymbol{u}'^2)]\,\boldsymbol{v}' \\ &\quad + [b^*\,(\boldsymbol{u}'^2) - (f^* - \xi)\,(\boldsymbol{u}' \cdot \boldsymbol{v}')]\,\boldsymbol{w}'. \end{aligned} \tag{3.8f}$$

Let us now write [compare (2.24)]

$$D'_1 = |(\boldsymbol{B} - \boldsymbol{C}) \;\; \boldsymbol{v}' \;\; \boldsymbol{w}'|, \quad D'_2 = |(\boldsymbol{B} - \boldsymbol{C}) \;\; \boldsymbol{w}' \;\; \boldsymbol{u}'|, \quad D'_3 = |(\boldsymbol{B} - \boldsymbol{C}) \;\; \boldsymbol{u}' \;\; \boldsymbol{v}'|; \tag{3.9a}$$

$$E'_1 = |(\boldsymbol{C} - \boldsymbol{A}) \;\; \boldsymbol{v}' \;\; \boldsymbol{w}'|, \quad E'_2 = |(\boldsymbol{C} - \boldsymbol{A}) \;\; \boldsymbol{w}' \;\; \boldsymbol{u}'|, \quad E'_3 = |(\boldsymbol{C} - \boldsymbol{A}) \;\; \boldsymbol{u}' \;\; \boldsymbol{v}'|; \tag{3.9b}$$

$$F'_1 = |(\boldsymbol{A} - \boldsymbol{B}) \;\; \boldsymbol{v}' \;\; \boldsymbol{w}'|, \quad F'_2 = |(\boldsymbol{A} - \boldsymbol{B}) \;\; \boldsymbol{w}' \;\; \boldsymbol{u}'|, \quad F'_3 = |(\boldsymbol{A} - \boldsymbol{B}) \;\; \boldsymbol{u}' \;\; \boldsymbol{v}'|; \tag{3.9c}$$

and, similarly,

$$D'_4 = (\boldsymbol{B} - \boldsymbol{C}) \cdot \boldsymbol{u}', \quad D'_5 = (\boldsymbol{B} - \boldsymbol{C}) \cdot \boldsymbol{v}', \quad D'_6 = (\boldsymbol{B} - \boldsymbol{C}) \cdot \boldsymbol{w}', \tag{3.10a}$$

$$E'_4 = (\boldsymbol{C} - \boldsymbol{A}) \cdot \boldsymbol{u}', \quad E'_5 = (\boldsymbol{C} - \boldsymbol{A}) \cdot \boldsymbol{v}', \quad E'_6 = (\boldsymbol{C} - \boldsymbol{A}) \cdot \boldsymbol{w}', \tag{3.10b}$$

$$F'_4 = (\mathbf{A} - \mathbf{B}) \,.\, \boldsymbol{u}', \quad F'_5 = (\mathbf{A} - \mathbf{B}) \,.\, \boldsymbol{v}', \quad F'_6 = (\mathbf{A} - \mathbf{B}) \,.\, \boldsymbol{w}'. \qquad (3.10c)$$

Then (3.2c) - (3.2e) yield that

$$\begin{aligned}
h_{11}\Big\{&\big[b^* \boldsymbol{w}'^2 + c^* \boldsymbol{v}'^2 - d^* \boldsymbol{v}' . \boldsymbol{w}'\big] D'_4 \\
&+ \big[\eta\, \boldsymbol{w}' . \boldsymbol{u}' + \zeta\, \boldsymbol{v}' . \boldsymbol{w}' - c^* \boldsymbol{u}' . \boldsymbol{v}' - (f^* - \xi)\, \boldsymbol{w}'^2\big] D'_5 \\
&+ \big[(d^* - \eta)\, \boldsymbol{u}' . \boldsymbol{v}' + (f^* - \xi)\, \boldsymbol{v}' . \boldsymbol{w}' - b^* \boldsymbol{w}' . \boldsymbol{u}' - \zeta\, \boldsymbol{v}'^2\big] D'_6\Big\} \\
+ h_{12}\Big\{&\big[(d^* - \eta)\, \boldsymbol{w}' . \boldsymbol{u}' + (e^* - \zeta)\, \boldsymbol{v}' . \boldsymbol{w}' - c^* \boldsymbol{u}' . \boldsymbol{v}' - \xi\, \boldsymbol{w}'^2\big] D'_4 \\
&+ \big[a^* \boldsymbol{w}'^2 + c^* \boldsymbol{u}'^2 - e^* \boldsymbol{w}' . \boldsymbol{u}'\big] D'_5 \\
&+ \big[\xi\, \boldsymbol{w}' . \boldsymbol{u}' + \zeta\, \boldsymbol{u}' . \boldsymbol{v}' - a^* \boldsymbol{v}' . \boldsymbol{w}' - (d^* - \eta)\, \boldsymbol{u}'^2\big] D'_6\Big\} \\
+ h_{13}\Big\{&\big[\xi\, \boldsymbol{v}' . \boldsymbol{w}' + \eta\, \boldsymbol{u}' . \boldsymbol{v}' - b^* \boldsymbol{w}' . \boldsymbol{u}' - (e^* - \zeta)\, \boldsymbol{v}'^2\big] D'_4 \\
&+ \big[(e^* - \zeta)\, \boldsymbol{u}' . \boldsymbol{v}' + (f^* - \xi)\, \boldsymbol{w}' . \boldsymbol{u}' - a^* \boldsymbol{v}' . \boldsymbol{w}' - \eta\, \boldsymbol{u}'^2\big] D'_5 \\
&+ \big[a^* \boldsymbol{v}'^2 + b^* \boldsymbol{u}'^2 - f^* \boldsymbol{u}' . \boldsymbol{v}'\big] D'_6\Big\} \\
&= \Delta(h_{12} - h_{13}) - (h_{11}D'_1 + h_{12}D'_2 + h_{13}D'_3), \qquad (3.11a)
\end{aligned}$$

$$\begin{aligned}
h_{21}\Big\{&\big[b^* \boldsymbol{w}'^2 + c^* \boldsymbol{v}'^2 - d^* \boldsymbol{v}' . \boldsymbol{w}'\big] E'_4 \\
&+ \big[\eta\, \boldsymbol{w}' . \boldsymbol{u}' + \zeta\, \boldsymbol{v}' . \boldsymbol{w}' - c^* \boldsymbol{u}' . \boldsymbol{v}' - (f^* - \xi)\, \boldsymbol{w}'^2\big] E'_5 \\
&+ \big[(d^* - \eta)\, \boldsymbol{u}' . \boldsymbol{v}' + (f^* - \xi)\, \boldsymbol{v}' . \boldsymbol{w}' - b^* \boldsymbol{w}' . \boldsymbol{u}' - \zeta\, \boldsymbol{v}'^2\big] E'_6\Big\} \\
+ h_{22}\Big\{&\big[(d^* - \eta)\, \boldsymbol{w}' . \boldsymbol{u}' + (e^* - \zeta)\, \boldsymbol{v}' . \boldsymbol{w}' - c^* \boldsymbol{u}' . \boldsymbol{v}' - \xi\, \boldsymbol{w}'^2\big] E'_4 \\
&+ \big[a^* \boldsymbol{w}'^2 + c^* \boldsymbol{u}'^2 - e^* \boldsymbol{w}' . \boldsymbol{u}'\big] E'_5 \\
&+ \big[\xi\, \boldsymbol{w}' . \boldsymbol{u}' + \zeta\, \boldsymbol{u}' . \boldsymbol{v}' - a^* \boldsymbol{v}' . \boldsymbol{w}' - (d^* - \eta)\, \boldsymbol{u}'^2\big] E'_6\Big\}
\end{aligned}$$

(CONTINUED...)

$$+ h_{23}\Big\{\big[\xi\, \boldsymbol{v}' \,.\, \boldsymbol{w}' + \eta\, \boldsymbol{u}' \,.\, \boldsymbol{v}' - b^*\, \boldsymbol{w}' \,.\, \boldsymbol{u}' - (e^* - \zeta)\, \boldsymbol{v}'^2\big]\, E'_4$$

$$+ \big[(e^* - \zeta)\, \boldsymbol{u}' \,.\, \boldsymbol{v}' + (f^* - \xi)\, \boldsymbol{w}' \,.\, \boldsymbol{u}' - a^*\, \boldsymbol{v}' \,.\, \boldsymbol{w}' - \eta\, \boldsymbol{u}'^2\big]\, E'_5$$

$$+ \big[a^*\, \boldsymbol{v}'^2 + b^*\, \boldsymbol{u}'^2 - f^*\, \boldsymbol{u}' \,.\, \boldsymbol{v}'\big]\, E'_6\Big\}$$

$$= \Delta(h_{23} - h_{21}) - (h_{21}E'_1 + h_{22}E'_2 + h_{23}E'_3), \quad (3.11\text{b})$$

$$h_{31}\Big\{\big[b^*\, \boldsymbol{w}'^2 + c^*\, \boldsymbol{v}'^2 - d^*\, \boldsymbol{v}' \,.\, \boldsymbol{w}'\big]\, F_4$$

$$+ \big[\eta\, \boldsymbol{w}' \,.\, \boldsymbol{u}' + \zeta\, \boldsymbol{v}' \,.\, \boldsymbol{w}' - c^*\, \boldsymbol{u}' \,.\, \boldsymbol{v}' - (f^* - \xi)\, \boldsymbol{w}'^2\big]\, F_5$$

$$+ \big[(d^* - \eta)\, \boldsymbol{u}' \,.\, \boldsymbol{v}' + (f^* - \xi)\, \boldsymbol{v}' \,.\, \boldsymbol{w}' - b^*\, \boldsymbol{w}' \,.\, \boldsymbol{u}' - \zeta\, \boldsymbol{v}'^2\big]\, F_6\Big\}$$

$$+ h_{32}\Big\{\big[(d^* - \eta)\, \boldsymbol{w}' \,.\, \boldsymbol{u}' + (e^* - \zeta)\, \boldsymbol{v}' \,.\, \boldsymbol{w}' - c^*\, \boldsymbol{u}' \,.\, \boldsymbol{v}' - \xi\, \boldsymbol{w}'^2\big]\, F_4$$

$$+ \big[a^*\, \boldsymbol{w}'^2 + c^*\, \boldsymbol{u}'^2 - e^*\, \boldsymbol{w}' \,.\, \boldsymbol{u}'\big]\, F_5$$

$$+ \big[\xi\, \boldsymbol{w}' \,.\, \boldsymbol{u}' + \zeta\, \boldsymbol{u}' \,.\, \boldsymbol{v}' - a^*\, \boldsymbol{v}' \,.\, \boldsymbol{w}' - (d^* - \eta)\, \boldsymbol{u}'^2\big]\, F_6\Big\}$$

$$+ h_{33}\Big\{\big[\xi\, \boldsymbol{v}' \,.\, \boldsymbol{w}' + \eta\, \boldsymbol{u}' \,.\, \boldsymbol{v}' - b^*\, \boldsymbol{w}' \,.\, \boldsymbol{u}' - (e^* - \zeta)\, \boldsymbol{v}'^2\big]\, F_4$$

$$+ \big[(e^* - \zeta)\, \boldsymbol{u}' \,.\, \boldsymbol{v}' + (f^* - \xi)\, \boldsymbol{w}' \,.\, \boldsymbol{u}' - a^*\, \boldsymbol{v}' \,.\, \boldsymbol{w}' - \eta\, \boldsymbol{u}'^2\big]\, F_5$$

$$+ \big[a^*\, \boldsymbol{v}'^2 + b^*\, \boldsymbol{u}'^2 - f^*\, \boldsymbol{u}' \,.\, \boldsymbol{v}'\big]\, F_6\Big\}$$

$$= \Delta(h_{31} - h_{32}) - (h_{31}F_1 + h_{32}F_2 + h_{33}F_3). \quad (3.11\text{c})$$

These rather lengthy equations are very simply solved. They take the form

$$\boldsymbol{M}\begin{bmatrix}\xi \\ \eta \\ \zeta\end{bmatrix} = \boldsymbol{g}, \quad (3.12)$$

where

$$M_{11} = [h_{13}\,\boldsymbol{v}'\cdot\boldsymbol{w}' - h_{12}\,\boldsymbol{w}'^2]D'_4 + [h_{11}\,\boldsymbol{w}'^2 - h_{13}\,\boldsymbol{w}'\cdot\boldsymbol{u}']D'_5 + [h_{12}\,\boldsymbol{w}'\cdot\boldsymbol{u}' - h_{11}\,\boldsymbol{v}'\cdot\boldsymbol{w}']D'_6, \tag{3.13a}$$

$$M_{12} = [h_{13}\,\boldsymbol{u}'\cdot\boldsymbol{v}' - h_{12}\,\boldsymbol{w}'\cdot\boldsymbol{u}']D'_4 + [h_{11}\,\boldsymbol{w}'\cdot\boldsymbol{u}' - h_{13}\,\boldsymbol{u}'^2]D'_5 + [h_{12}\,\boldsymbol{u}'^2 - h_{11}\,\boldsymbol{u}'\cdot\boldsymbol{v}']D'_6, \tag{3.13b}$$

$$M_{13} = [h_{13}\,\boldsymbol{v}'^2 - h_{12}\,\boldsymbol{v}'\cdot\boldsymbol{w}']D'_4 + [h_{11}\,\boldsymbol{v}'\cdot\boldsymbol{w}' - h_{13}\,\boldsymbol{u}'\cdot\boldsymbol{v}']D'_5 + [h_{12}\,\boldsymbol{u}'\cdot\boldsymbol{v}' - h_{11}\,\boldsymbol{v}'^2]D'_6, \tag{3.13c}$$

$$M_{21} = [h_{23}\,\boldsymbol{v}'\cdot\boldsymbol{w}' - h_{22}\,\boldsymbol{w}'^2]E'_4 + [h_{21}\,\boldsymbol{w}'^2 - h_{23}\,\boldsymbol{w}'\cdot\boldsymbol{u}']E'_5 + [h_{22}\,\boldsymbol{w}'\cdot\boldsymbol{u}' - h_{21}\,\boldsymbol{v}'\cdot\boldsymbol{w}']E'_6, \tag{3.13d}$$

$$M_{22} = [h_{23}\,\boldsymbol{u}'\cdot\boldsymbol{v}' - h_{22}\,\boldsymbol{w}'\cdot\boldsymbol{u}']E'_4 + [h_{21}\,\boldsymbol{w}'\cdot\boldsymbol{u}' - h_{23}\,\boldsymbol{u}'^2]E'_5 + [h_{22}\,\boldsymbol{u}'^2 - h_{21}\,\boldsymbol{u}'\cdot\boldsymbol{v}']E'_6, \tag{3.13e}$$

$$M_{23} = [h_{23}\,\boldsymbol{v}'^2 - h_{22}\,\boldsymbol{v}'\cdot\boldsymbol{w}']E'_4 + [h_{21}\,\boldsymbol{v}'\cdot\boldsymbol{w}' - h_{23}\,\boldsymbol{u}'\cdot\boldsymbol{v}']E'_5 + [h_{22}\,\boldsymbol{u}'\cdot\boldsymbol{v}' - h_{21}\,\boldsymbol{v}'^2]E'_6, \tag{3.13f}$$

$$M_{31} = [h_{33}\,\boldsymbol{v}'\cdot\boldsymbol{w}' - h_{32}\,\boldsymbol{w}'^2]F'_4 + [h_{31}\,\boldsymbol{w}'^2 - h_{33}\,\boldsymbol{w}'\cdot\boldsymbol{u}']F'_5 + [h_{32}\,\boldsymbol{w}'\cdot\boldsymbol{u}' - h_{31}\,\boldsymbol{v}'\cdot\boldsymbol{w}']F'_6, \tag{3.13g}$$

$$M_{32} = [h_{33}\,\boldsymbol{u}'\cdot\boldsymbol{v}' - h_{32}\,\boldsymbol{w}'\cdot\boldsymbol{u}']F'_4 + [h_{31}\,\boldsymbol{w}'\cdot\boldsymbol{u}' - h_{33}\,\boldsymbol{u}'^2]F'_5 + [h_{32}\,\boldsymbol{u}'^2 - h_{31}\,\boldsymbol{u}'\cdot\boldsymbol{v}']F'_6, \tag{3.13h}$$

$$M_{33} = [h_{33}\,\boldsymbol{v}'^2 - h_{32}\,\boldsymbol{v}'\cdot\boldsymbol{w}']F'_4 + [h_{31}\,\boldsymbol{v}'\cdot\boldsymbol{w}' - h_{33}\,\boldsymbol{u}'\cdot\boldsymbol{v}']F'_5 + [h_{32}\,\boldsymbol{u}'\cdot\boldsymbol{v}' - h_{31}\,\boldsymbol{v}'^2]F'_6, \tag{3.13i}$$

and

$$
\begin{aligned}
g_1 = {} & \Delta(h_{12} - h_{13}) - (h_{11}D'_1 + h_{12}D'_2 + h_{13}D'_3) \\
& - h_{11}\Big\{\big[b^* \mathbf{w}'^2 + c^* \mathbf{v}'^2 - d^* \mathbf{v}' \,.\, \mathbf{w}'\big] D'_4 \\
& \qquad + \big[- c^* \mathbf{u}' \,.\, \mathbf{v}' - f^* \mathbf{w}'^2\big] D'_5 \\
& \qquad + \big[d^* \mathbf{u}' \,.\, \mathbf{v}' + f^* \mathbf{v}' \,.\, \mathbf{w}' - b^* \mathbf{w}' \,.\, \mathbf{u}'\big] D'_6\Big\} \\
& - h_{12}\Big\{\big[d^* \mathbf{w}' \,.\, \mathbf{u}' + e^* \mathbf{v}' \,.\, \mathbf{w}' - c^* \mathbf{u}' \,.\, \mathbf{v}'\big] D'_4 \\
& \qquad + \big[a^* \mathbf{w}^2 + c^* \mathbf{u}'^2 - e^* \mathbf{w}' \,.\, \mathbf{u}'\big] D'_5 \\
& \qquad + \big[- a^* \mathbf{v}' \,.\, \mathbf{w}' - d^* \mathbf{u}'^2\big] D'_6\Big\} \\
& - h_{13}\Big\{\big[- b^* \mathbf{w}' \,.\, \mathbf{u}' - e^* \mathbf{v}'^2\big] D'_4 \\
& \qquad + \big[e^* \mathbf{u}' \,.\, \mathbf{v}' + f^* \mathbf{w}' \,.\, \mathbf{u}' - a^* \mathbf{v}' \,.\, \mathbf{w}'\big] D'_5 \\
& \qquad + \big[a^* \mathbf{v}'^2 + b^* \mathbf{u}'^2 - f^* \mathbf{u}' \,.\, \mathbf{v}'\big] D'_6\Big\}. \qquad (3.14a)
\end{aligned}
$$

$$
\begin{aligned}
g_2 = {} & \Delta(h_{23} - h_{21}) - (h_{21}E'_1 + h_{22}E'_2 + h_{23}E'_3) \\
& - h_{21}\Big\{\big[b^* \mathbf{w}'^2 + c^* \mathbf{v}'^2 - d^* \mathbf{v}' \,.\, \mathbf{w}'\big] E'_4 \\
& \qquad + \big[- c^* \mathbf{u}' \,.\, \mathbf{v}' - f^* \mathbf{w}'^2\big] E'_5 \\
& \qquad + \big[d^* \mathbf{u}' \,.\, \mathbf{v}' + f^* \mathbf{v}' \,.\, \mathbf{w}' - b^* \mathbf{w}' \,.\, \mathbf{u}'\big] E'_6\Big\} \\
& - h_{22}\Big\{\big[d^* \mathbf{w}' \,.\, \mathbf{u}' + e^* \mathbf{v}' \,.\, \mathbf{w}' - c^* \mathbf{u}' \,.\, \mathbf{v}'\big] E'_4 \\
& \qquad + \big[a^* \mathbf{w}'^2 + c^* \mathbf{u}'^2 - e^* \mathbf{w}' \,.\, \mathbf{u}'\big] E'_5 \\
& \qquad + \big[- a^* \mathbf{v}' \,.\, \mathbf{w}' - d^* \mathbf{u}'^2\big] E'_6\Big\} \\
& - h_{23}\Big\{\big[- b^* \mathbf{w}' \,.\, \mathbf{u}' - e^* \mathbf{v}'^2\big] E'_4 \\
& \qquad + \big[e^* \mathbf{u}' \,.\, \mathbf{v}' + f^* \mathbf{w}' \,.\, \mathbf{u}' - a^* \mathbf{v}' \,.\, \mathbf{w}'\big] E'_5 \\
& \qquad + \big[a^* \mathbf{v}'^2 + b^* \mathbf{u}'^2 - f^* \mathbf{u}' \,.\, \mathbf{v}'\big] E'_6\Big\}. \qquad (3.14b)
\end{aligned}
$$

$$
\begin{aligned}
g_3 = {} & \Delta(h_{31} - h_{32}) - (h_{31}F'_1 + h_{32}F'_2 + h_{33}F'_3) \\
& - h_{31}\Big\{\left[b^* \mathbf{w}'^2 + c^* \mathbf{v}'^2 - d^* \mathbf{v}' \cdot \mathbf{w}'\right] F'_4 \\
& \qquad + \left[- c^* \mathbf{u}' \cdot \mathbf{v}' - f^* \mathbf{w}'^2\right] F'_5 \\
& \qquad + \left[d^* \mathbf{u}' \cdot \mathbf{v}' + f^* \mathbf{v}' \cdot \mathbf{w}' - b^* \mathbf{w}' \cdot \mathbf{u}'\right] F'_6\Big\} \\
& - h_{32}\Big\{\left[d^* \mathbf{w}' \cdot \mathbf{u}' + e^* \mathbf{v}' \cdot \mathbf{w}' - c^* \mathbf{u}' \cdot \mathbf{v}'\right] F'_4 \\
& \qquad + \left[a^* \mathbf{w}'^2 + c^* \mathbf{u}'^2 - e^* \mathbf{w}' \cdot \mathbf{u}'\right] F'_5 \\
& \qquad + \left[- a^* \mathbf{v}' \cdot \mathbf{w}' - d^* \mathbf{u}'^2\right] F'_6\Big\} \\
& - h_{33}\Big\{\left[- b^* \mathbf{w}' \cdot \mathbf{u}' - e^* \mathbf{v}'^2\right] F'_4 \\
& \qquad + \left[e^* \mathbf{u}' \cdot \mathbf{v}' + f^* \mathbf{w}' \cdot \mathbf{u}' - a^* \mathbf{v}' \cdot \mathbf{w}'\right] F'_5 \\
& \qquad + \left[a^* \mathbf{v}'^2 + b^* \mathbf{u}'^2 - f^* \mathbf{u}' \cdot \mathbf{v}'\right] F'_6\Big\}.
\end{aligned}
\tag{3.14c}
$$

It will be noticed that the computation of these coefficients is greatly facilitated by the repetition of the same combinations of parameters. In particular, if we write

$$
\left.
\begin{aligned}
P'_1 &= b^* \mathbf{w}'^2 + c^* \mathbf{v}'^2 - d^* \mathbf{v}' \cdot \mathbf{w}' \\
P'_2 &= - c^* \mathbf{u}' \cdot \mathbf{v}' - f^* \mathbf{w}'^2 \\
P'_3 &= d^* \mathbf{u}' \cdot \mathbf{v}' + f^* \mathbf{v}' \cdot \mathbf{w}' - b^* \mathbf{w}' \cdot \mathbf{u}'
\end{aligned}
\right\},
\tag{3.15a}
$$

$$
\left.
\begin{aligned}
Q'_1 &= d^* \mathbf{w}' \cdot \mathbf{u}' + e^* \mathbf{v}' \cdot \mathbf{w}' - c^* \mathbf{u}' \cdot \mathbf{v}' \\
Q'_2 &= - a^* \mathbf{w}'^2 + c^* \mathbf{u}'^2 - e^* \mathbf{w}' \cdot \mathbf{u}' \\
Q'_3 &= - a^* \mathbf{v}' \cdot \mathbf{w}' - d^* \mathbf{u}'^2
\end{aligned}
\right\},
\tag{3.15b}
$$

$$
\left.
\begin{aligned}
R'_1 &= - b^* \mathbf{w}' \cdot \mathbf{u}' - e^* \mathbf{v}'^2 \\
R'_2 &= e^* \mathbf{u}' \cdot \mathbf{v}' + f^* \mathbf{w}' \cdot \mathbf{u}' - a^* \mathbf{v}' \cdot \mathbf{w}' \\
R'_3 &= a^* \mathbf{v}'^2 + b^* \mathbf{u}'^2 - f^* \mathbf{u}' \cdot \mathbf{v}'
\end{aligned}
\right\};
\tag{3.15c}
$$

then (3.14) becomes

$$
\begin{aligned}
g_1 = {} & \Delta(h_{12} - h_{13}) - (h_{11}D'_1 + h_{12}D'_2 + h_{13}D'_3) \\
& - h_{11}(P'_1\, D'_4 + P'_2\, D'_5 + P'_3\, D'_6) \\
& - h_{12}(Q'_1\, D'_4 + Q'_2\, D'_5 + Q'_3\, D'_6) \\
& - h_{13}(R'_1\, D'_4 + R'_2\, D'_5 + R'_3\, D'_6),
\end{aligned}
\tag{3.16a}
$$

$$
\begin{aligned}
g_2 = {} & \Delta(h_{23} - h_{21}) - (h_{21}E'_1 + h_{22}E'_2 + h_{23}E'_3) \\
& - h_{21}(P'_1\, E'_4 + P'_2\, E'_5 + P'_3\, E'_6) \\
& - h_{22}(Q'_1\, E'_4 + Q'_2\, E'_5 + Q'_3\, E'_6) \\
& - h_{23}(R'_1\, E'_4 + R'_2\, E'_5 + R'_3\, E'_6),
\end{aligned}
\tag{3.16b}
$$

$$
\begin{aligned}
g_3 = {} & \Delta(h_{31} - h_{32}) - (h_{31}F'_1 + h_{32}F'_2 + h_{33}F'_3) \\
& - h_{31}(P'_1\, F'_4 + P'_2\, F'_5 + P'_3\, F'_6) \\
& - h_{32}(Q'_1\, F'_4 + Q'_2\, F'_5 + Q'_3\, F'_6) \\
& - h_{33}(R'_1\, F'_4 + R'_2\, F'_5 + R'_3\, F'_6).
\end{aligned}
\tag{3.16c}
$$

Thereafter, the solution of the system of three equations in three unknowns is easy to perform.

Of course, as is well known, Newton's method is asymptotically at least *quadratically convergent*. This is a very fast rate of convergence, highly satisfactory for most purposes. [See §A4 of the Appendix for more detail; we shall return to this matter in §5.]

4. Operations Count

Let us now suppose that the equations (2.4) and (2.25) are reduced to linear forms (3.5), then (3.11), and finally (3.12) – (3.16); *via* (2.5), (2.22), (3.1), (3.3), (3.4), (3.7), (3.9), (3.10), (A24), and (A26); and solved for $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ by applying Newtonian iterations as explained above. We seek to establish the time required to perform

the necessary operations leading up to the process of solution and, from that solution, back to the position vector $\boldsymbol{s}$; as well as the time required by each iteration of Newton's method. Since different computers have different timing characteristics for floating-point operations (FLOPs), and since these usually take considerably longer than elementary operations, such as conditional jumps, RAM read and store commands, integer count incrementations, sign-changes, shifts, etc. (which, anyway, are not excessively numerous in the kind of computation considered here); all we need to do is to count, on the one hand, all the *multiplications and divisions* (M/D), and, on the other hand, all the *additions and subtractions* (A/S), required by the computations being considered. The details of the required derivations are assembled in §A3 of the Appendix.

Before the readings begin, we need to establish the values of a, b, c, d, e, and f in (2.1); and, by (2.22), these yield the value of Δ. By (A33), the FLOP-count for computing Δ is $\mathcal{D}$ = 9 M/D + 5 A/S. We also need 2Δ, which takes 1 A/S more.

Now, each time that we take a set of location-readings, we identify three LED landmark position vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, and a matrix $\boldsymbol{K}$ of direction-vectors relative to the headmounted coordinate system [see (2.5) – (2.8)]. Some computation may be necessary to convert raw directional data from the three cameras into the nine components k_{ij}, but we shall assume that this is done very quickly. We also require the vectors $\boldsymbol{B} - \boldsymbol{C}$, $\boldsymbol{C} - \boldsymbol{A}$, and $\boldsymbol{A} - \boldsymbol{B}$, for our computations; these three vector subtractions take a trivial 9 A/S to obtain.

The process of inverting $\boldsymbol{K}$ to obtain its reciprocal $\boldsymbol{H}$ takes, by (A37), $\mathcal{R}$ = 27 M/D + 18 A/S; and $|\boldsymbol{K}|$ is derived, by (A38), as a by-product of this calculation in only $\mathcal{D}_{\text{with } \mathcal{R}}$ = 2 M/D more. Then, we need the quantities $\Delta(h_{12} - h_{23})$, $\Delta(h_{12} - h_{23})$, and $\Delta(h_{12} - h_{23})$ in applying (3.16) to the iterations; these take an additional 3 M/D + 3 A/S.

Once we have performed the iterations until a satisfactory error-estimate is obtained, we must use our final iterates $\boldsymbol{u}^*$, $\boldsymbol{v}^*$, and $\boldsymbol{w}^*$, say, to obtain $\boldsymbol{s}$. To do so, we must first compute the vectors $\boldsymbol{p}^*$, $\boldsymbol{q}^*$, and $\boldsymbol{r}^*$, using (2.5); by (A41), this takes $\mathcal{M}$ = 27 M/D + 18 A/S. We also need the vectors $\boldsymbol{B} - \boldsymbol{v}^*$, $\boldsymbol{C} - \boldsymbol{w}^*$, and $\boldsymbol{A} - \boldsymbol{u}^*$; this takes 9 A/S in all. Then we can proceed in two ways.

(a) We can compute the approximation $\boldsymbol{s}^*$ directly from (2.14). The computation of the common denominator $|\,\boldsymbol{p}^* \quad \boldsymbol{q}^* \quad \boldsymbol{r}^*\,|$ takes only 1 multiplication, because of (2.23). By (A15) and (A33), the scalar numerators take $3\mathcal{D}$ = 27 M/D + 15 A/S, to form the triple scalar products; then 3 divisions yield the scalar coefficients, and

3(3 M/D + 2A/S) complete the evaluation. Thus, in all, we need 40 M/D + 21 A/S.

(b) We can solve (2.11), taking advantage of the simple structure of the equations. First, we respectively eliminate λ and μ from the first pair of equations in (2.11a) and (2.11b), yielding

$$p_2^* s_1^* - p_1^* s_2^* = p_2^*(A_1 - u_1^*) - p_1^*(A_2 - u_2^*),$$
$$q_2^* s_1^* - q_1^* s_2^* = q_2^*(B_1 - v_1^*) - q_1^*(B_2 - v_2^*);$$

whence

$$s_1^* = \frac{p_2^* q_1^*(A_1 - u_1^*) - p_1^* q_2^*(B_1 - v_1^*) - p_1^* q_1^*(A_2 - B_2 + v_2^* - u_2^*)}{p_2^* q_1^* - p_1^* q_2^*},$$
$$s_2^* = \frac{p_1^* q_2^*(A_2 - u_2^*) - p_2^* q_1^*(B_2 - v_2^*) - p_2^* q_2^*(A_1 - B_1 + v_1^* - u_1^*)}{p_1^* q_2^* - p_2^* q_1^*}; \tag{4.1}$$

and then we eliminate v from the last two equations in (2.11c), yielding

$$r_3^* s_2^* - r_2^* s_3^* = r_3^*(C_2 - w_2^*) - r_2^*(C_3 - w_3^*),$$

whence
$$s_3^* = C_3 - w_3^* + \frac{r_3^*}{r_2^*}(s_2^* - C_2 + w_2^*). \tag{4.2}$$

[The temptation must be resisted, to substitute the second algebraic formula in (4.1) into (4.2); this leads to more FLOPs!] An operations count on (4.1) and (4.2) shows that we need 14 M/D + 9 A/S in all. Clearly, the gain in using the second method is considerable.

Summing up, we see that the computations required at every location-reading require altogether

$$\mathcal{A} = 73\ \text{M/D} + 66\ \text{A/S}. \tag{4.3}$$

Now we turn to the computations entailed by each *iteration*. The equations to be solved are given by (3.12), with (3.13), (3.15), and (3.16). By (A40), it takes $\mathcal{G}$ = 17 M/D + 11 A/S to solve them for ξ, η, and ζ, once the coefficients are known.

To get the M_{ij}, we first need to calculate the six scalar products $\boldsymbol{u}'^2$, $\boldsymbol{v}'^2$, $\boldsymbol{w}'^2$, $\boldsymbol{v}' \,.\, \boldsymbol{w}'$, $\boldsymbol{w}' \,.\, \boldsymbol{u}'$, and $\boldsymbol{u}' \,.\, \boldsymbol{v}'$; this takes $6 \times$ (3 M/D + 2 A/S). Then, we require the nine coefficients D'_4, D'_5, D'_6, E'_4, E'_5, E'_6, F'_4, F'_5, and F'_6, defined in (3.10); this takes another $9 \times$ (3 M/D + 2 A/S). Finally, by (3.13), we require $9 \times$ (9 M/D + 5 A/S). The total is 126 M/D + 75 A/S.

To get the g_i, we must first compute the six parameters a^*, b^*, c^*, d^*, e^*, and f^*, by (3.3), taking 6 M/D + 6 A/S. Next, we need to compute the three vector products $\boldsymbol{v}' \wedge \boldsymbol{w}'$, $\boldsymbol{w}' \wedge \boldsymbol{u}'$, and $\boldsymbol{u}' \wedge \boldsymbol{v}'$; this takes, by (A31), $3\mathcal{V}$ = 18 M/D + 9 A/S. Then we need the eighteen coefficients D'_1, D'_2, D'_3, E'_1, E'_2, E'_3, F'_1, F'_2, and F'_3, defined in (3.9), and P'_1, P'_2, P'_3, Q'_1, Q'_2, Q'_3, R'_1, R'_2, and R'_3, defined in (3.15). The former take $9 \times$ (3 M/D + 2 A/S) = 27 M/D + 18 A/S; the latter take another 24 M/D + 15 A/S. Finally, by (3.16), we require $3 \times$ (15 M/D + 12 A/S). The total is 120 M/D + 84 A/S.

All in all, we need 246 M/D + 159 A/S to obtain ξ, η, and ζ, for one iteration. Next, we need to apply (3.6), with 27 M/D + 21 A/S, to get $\delta\boldsymbol{u}'$, $\delta\boldsymbol{v}'$, and $\delta\boldsymbol{w}'$; whence we finally get $\boldsymbol{u}''$, $\boldsymbol{v}''$, and $\boldsymbol{w}''$ in 9 A/S. Thus, the computations in every iteration of the Newton process described above require altogether

$$\mathcal{B} = 273 \text{ M/D} + 189 \text{ A/S}. \tag{4.4}$$

5. THE RESTRICTED SYSTEM

A considerable simplification is effected by restricting every iterate $\{\boldsymbol{u}^{(m)}, \boldsymbol{v}^{(m)}, \boldsymbol{w}^{(m)}\} = \{\boldsymbol{u}', \boldsymbol{v}', \boldsymbol{w}'\}$ to conform to the exact relations (2.4); so that

$$\boldsymbol{u}'^2 = a, \quad \boldsymbol{v}'^2 = b, \quad \boldsymbol{w}'^2 = c, \tag{5.1}$$

and
$$\boldsymbol{v}' \,.\, \boldsymbol{w}' = d, \quad \boldsymbol{w}' \,.\, \boldsymbol{u}' = e, \quad \boldsymbol{u}' \,.\, \boldsymbol{v}' = f. \tag{5.2}$$

Then, by (3.3),

$$a^* = b^* = c^* = d^* = e^* = f^* = 0; \tag{5.3}$$

and (3.6) becomes

$$\left.\begin{aligned} \delta \boldsymbol{u}' &= \xi\, \boldsymbol{w}' \wedge \boldsymbol{u}' - \zeta\, \boldsymbol{u}' \wedge \boldsymbol{v}' \\ \delta \boldsymbol{v}' &= \eta\, \boldsymbol{u}' \wedge \boldsymbol{v}' - \xi\, \boldsymbol{v}' \wedge \boldsymbol{w}' \\ \delta \boldsymbol{w}' &= \zeta\, \boldsymbol{v}' \wedge \boldsymbol{w}' - \eta\, \boldsymbol{w}' \wedge \boldsymbol{u}' \end{aligned}\right\}. \tag{5.4}$$

Thus, by (5.1) and (5.2), the equations (3.13) become

$$M_{11} = (dh_{13} - ch_{12})D'_4 + (ch_{11} - eh_{13})D'_5 + (eh_{12} - dh_{11})D'_6. \tag{5.5a}$$

$$M_{12} = (fh_{13} - eh_{12})D'_4 + (eh_{11} - ah_{13})D'_5 + (ah_{12} - fh_{11})D'_6. \tag{5.5b}$$

$$M_{13} = (bh_{13} - dh_{12})D'_4 + (dh_{11} - fh_{13})D'_5 + (fh_{12} - bh_{11})D'_6. \tag{5.5c}$$

$$M_{21} = (dh_{23} - ch_{22})E'_4 + (ch_{21} - eh_{23})E'_5 + (eh_{22} - dh_{21})E'_6. \tag{5.5d}$$

$$M_{22} = (fh_{23} - eh_{22})E'_4 + (eh_{21} - ah_{23})E'_5 + (ah_{22} - fh_{21})E'_6. \tag{5.5e}$$

$$M_{23} = (bh_{23} - dh_{22})E'_4 + (dh_{21} - fh_{23})E'_5 + (fh_{22} - bh_{21})E'_6. \tag{5.5f}$$

$$M_{31} = (dh_{33} - ch_{32})F'_4 + (ch_{31} - eh_{33})F'_5 + (eh_{32} - dh_{31})F'_6. \tag{5.5g}$$

$$M_{32} = (fh_{33} - eh_{32})F'_4 + (eh_{31} - ah_{33})F'_5 + (ah_{32} - fh_{31})F'_6. \tag{5.5h}$$

$$M_{33} = (bh_{33} - dh_{32})F'_4 + (dh_{31} - fh_{33})F'_5 + (fh_{32} - bh_{31})F'_6. \tag{5.5i}$$

and (3.14) becomes, by (5.3),

$$g_1 = \Delta(h_{12} - h_{13}) - (h_{11}D'_1 + h_{12}D'_2 + h_{13}D'_3). \tag{5.6a}$$

$$g_2 = \Delta(h_{23} - h_{21}) - (h_{21}E'_1 + h_{22}E'_2 + h_{23}E'_3). \tag{5.6b}$$

$$g_3 = \Delta(h_{31} - h_{32}) - (h_{31}F'_1 + h_{32}F'_2 + h_{33}F'_3). \tag{5.6c}$$

We can now analyze the reduced operations count, much as we did in §4. Before readings begin, we again need a, b, c, d, e, and f, from which we compute Δ by 9 M/D + 5 A/S. We do not need 2Δ, however, saving 1 A/S.

All the non-iterated computations needed to deal with each new set of location readings in the general case are still needed here; they take $\mathcal{A}$ = 73 M/D + 66 A/S. However, now we also need to compute the twenty-seven coefficients in (5.5), unfortunately all different, each of which takes 2 M/D + 1 A/S. The total is

$$\mathcal{A}_{\perp} = 127 \text{ M/D} + 93 \text{ A/S}. \tag{5.7}$$

For each *iteration*, the same form of equations (3.12) must still be solved, for ξ, η, and ζ; once the coefficients are known, this again takes 17 M/D + 11 A/S.

To get the M_{ij}, we again first need to calculate the nine coefficients D'_4, D'_5, D'_6, E'_4, E'_5, E'_6, F'_4, F'_5, and F'_6, and this takes $9 \times$ (3 M/D + 2 A/S); then, by (5.5), another $9 \times$ (3 M/D + 2 A/S). The total is thus 54 M/D + 36 A/S.

To get the g_i, we first need the nine coefficients D'_1, D'_2, D'_3, E'_1, E'_2, E'_3, F'_1, F'_2, and F'_3, and this takes $9 \times$ (3 M/D + 2 A/S); then, by (5.6), another $3 \times$ (3 M/D + 3 A/S) are needed. Therefore, in all, we need 107 M/D + 74 A/S to obtain ξ, η, and ζ, for one iteration. To get $\delta\boldsymbol{u}'$, $\delta\boldsymbol{v}'$, and $\delta\boldsymbol{w}'$ from (5.4) requires *only* 18 M/D + 9 A/S; whence we finally get $\boldsymbol{u}''$, $\boldsymbol{v}''$, and $\boldsymbol{w}''$ in 9 A/S. Thus, the computations in every iteration of the Newton process require

$$\mathcal{B}_{\perp} = 125 \text{ M/D} + 92 \text{ A/S}. \tag{5.8}$$

This is less than half of the count in (4.4).

6. ANTICIPATORY INITIALIZATION

In §3, we perfunctorily suggested that the initial approximation, actually $\{\boldsymbol{s}^{(0)}, \boldsymbol{u}^{(0)}, \boldsymbol{v}^{(0)}, \boldsymbol{w}^{(0)}\}$, should be "the solution obtained (by Newton's method) for the most recent set of observations, as the wearer of the headmounted system moves about the laboratory." With a little further reflection, we can improve on this. As the observer wearing the headmounted system moves about, he or she can only do so by continuing the previous (translational or rotational) motion or by

applying a new force or torque to overcome the (linear and angular) inertia of his or her body, head, and helmet. This indicates that, by well-known, elementary dynamic principles, the values, and first and second derivatives with respect to time, of all the components of $\boldsymbol{s}$, $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ will be continuous and differentiable; though the third derivatives may be discontinuous.

Suppose now that we have successive observations, made at equal time-intervals, which we shall denote by

$$\boldsymbol{R}^{[\tau]} = \{\boldsymbol{A}^{[\tau]}, \boldsymbol{B}^{[\tau]}, \boldsymbol{C}^{[\tau]}, \boldsymbol{K}^{[\tau]}\}, \quad \tau = 0, 1, 2, \ldots; \tag{6.1}$$

where $\boldsymbol{A}^{[\tau]}$, $\boldsymbol{B}^{[\tau]}$, and $\boldsymbol{C}^{[\tau]}$ are the positional vectors of the landmark LEDs, $\mathrm{A}^{[\tau]}$, $\mathrm{B}^{[\tau]}$, and $\mathrm{C}^{[\tau]}$, and $\boldsymbol{K}^{[\tau]}$ is the matrix of observed directions given by (2.5). After the application of the Newtonian process defined in §3 or §5 (or by direct solution), we can obtain the corresponding state, which we shall denote by

$$\mathbf{S}^{[\tau]} = \{\boldsymbol{s}^{[\tau]}, \boldsymbol{u}^{[\tau]}, \boldsymbol{v}^{[\tau]}, \boldsymbol{w}^{[\tau]}\}. \tag{6.2}$$

Let us suppose that we have obtained the states $\mathbf{S}^{[\tau]}$ for $\tau = 0$, 1, and 2, by the method selected above. Then we may anticipate, by the continuity and differentiability of the values and of the first and second derivatives with respect to time (and, therefore, with respect to τ, treated as a continuous variable), and by Taylor's theorem, that a good approximation to the state (and therefore *an excellent initial state for Newton's method*) for any $\tau \geq 3$ will be given by a quadratic fit to the three preceding states; that is, if we temporarily write $\tau_0 = \tau - 2$, then there are constants X, Y, and Z, such that

$$\mathbf{S}^{[\tau](0)} = X + Y(\tau - \tau_0) + Z(\tau - \tau_0)^2 = X + 2Y + 4Z, \tag{6.3}$$

where

$$\left.\begin{aligned} \mathbf{S}^{[\tau-1]} &= X + Y[(\tau - 1) - \tau_0] + Z[(\tau - 1) - \tau_0]^2 = X + Y + Z \\ \mathbf{S}^{[\tau-2]} &= X + Y[(\tau - 2) - \tau_0] + Z[(\tau - 2) - \tau_0]^2 = X \\ \mathbf{S}^{[\tau-3]} &= X + Y[(\tau - 3) - \tau_0] + Z[(\tau - 3) - \tau_0]^2 = X - Y + Z \end{aligned}\right\}. \tag{6.4}$$

This simplifies, after a little algebra, to yield

$$\left.\begin{aligned} X &= \mathbf{S}^{[\tau-2]} \\ Y &= \frac{\mathbf{S}^{[\tau-1]} - \mathbf{S}^{[\tau-3]}}{2} \\ Z &= \frac{\mathbf{S}^{[\tau-1]} - 2\,\mathbf{S}^{[\tau-2]} + \mathbf{S}^{[\tau-3]}}{2} \end{aligned}\right\}; \tag{6.5}$$

whence (6.3) becomes

$$\mathbf{S}^{[\tau](0)} = 3\mathbf{S}^{[\tau-1]} - 3\mathbf{S}^{[\tau-2]} + \mathbf{S}^{[\tau-3]}. \tag{6.6}$$

The use of this initial state; that is, more precisely, of

$$\left.\begin{aligned} \boldsymbol{s}^{[\tau](0)} &= 3\boldsymbol{s}^{[\tau-1]} - 3\boldsymbol{s}^{[\tau-2]} + \boldsymbol{s}^{[\tau-3]} \\ \boldsymbol{u}^{[\tau](0)} &= 3\boldsymbol{u}^{[\tau-1]} - 3\boldsymbol{u}^{[\tau-2]} + \boldsymbol{u}^{[\tau-3]} \\ \boldsymbol{v}^{[\tau](0)} &= 3\boldsymbol{v}^{[\tau-1]} - 3\boldsymbol{v}^{[\tau-2]} + \boldsymbol{v}^{[\tau-3]} \\ \boldsymbol{w}^{[\tau](0)} &= 3\boldsymbol{w}^{[\tau-1]} - 3\boldsymbol{w}^{[\tau-2]} + \boldsymbol{w}^{[\tau-3]} \end{aligned}\right\}; \tag{6.7}$$

will ensure a super-fast convergence by Newton's method to the next state $\mathbf{S}^{[\tau]}$, using the formulæ already developed above.

7. CONCLUSIONS

We have seen that the equations for the position and orientation of the general three-camera headmounted location system—(2.4), (2.14), and (2.25), with (2.5) - (2.8), (2.21), (2.23), and (2.24)—can be solved by using a method of Newton's type. It is advantageous to restrict the method to require (5.1) and (5.2), when the increments in the iterates are given by (5.4); then the (3×3) system (3.12) must be solved, with coefficients given by (5.5) and (5.6).

By (5.7) and (5.8), we see that each set of location readings requires, if we perform m^* iterations in all to achieve a given accuracy,

$$\mathcal{A}_\perp + m^* \mathcal{B}_\perp = (127 + 125m^*)\ \text{M/D} + (93 + 92m^*)\ \text{A/S}; \tag{7.1}$$

or, let us say, approximately,

$$(m^* + 1)(127 \text{ M/D} + 93 \text{ A/S}). \tag{7.2}$$

It is rather difficult to estimate directly the number m^* of iterations needed. However, by the formulæ (A54) and (A55), we do know that—asymptotically as the errors tend to zero—if the improvement in any error norm (such as the largest absolute error among the components of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, and $\mathbf{s}$), in one iteration, is by a factor φ; then the improvement in the next iteration will be by the factor φ^2. [In the one-dimensional case, by (A48), we know that there is a constant C, such that, as $\varepsilon^{(m)} \to 0$,

$$\varepsilon^{(m+1)} \sim C\,\varepsilon^{(m)2}; \tag{7.3}$$

whence

$$\frac{\varepsilon^{(m+1)}}{\varepsilon^{(m)}} \sim C\,\varepsilon^{(m)}, \tag{7.4}$$

and therefore

$$\frac{\varepsilon^{(m+2)}}{\varepsilon^{(m+1)}} \sim C\,\varepsilon^{(m+1)} \sim C^2\,\varepsilon^{(m)2}$$

$$\sim \left[\frac{\varepsilon^{(m+1)}}{\varepsilon^{(m)}}\right]^2. \tag{7.5}$$

In the multi-dimensional case, the argument is similar.] One way of describing this is to say that, *in each iteration, one gains twice as many significant digits as in the previous iteration.* Thus, a relatively short experiment will indicate the number of iterations needed to achieve the required accuracy. We note that the greatest change in the orientation of the headmounted system, from reading to reading, will be of the order of $1° = \frac{2\pi}{360} < 0.02$ radian. Thus, the initial iterate $\{\boldsymbol{u}^{(0)}, \boldsymbol{v}^{(0)}, \boldsymbol{w}^{(0)}\}$ will differ from the true answer $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ by a relative error not exceeding 2%. If we apply the *anticipatory initialization* outlined in §6 above, we may expect a much better initialization, and therefore an even faster convergence to the required accuracy.

As a final guiding remark, we may point out that we have assumed 300 readings per second. For real-time operation, this gives us $\frac{1}{300}$ second = 3,333 μsec. to perform the computations in (7.2). If we suppose that we are using a typical 1 MFLOPS [1 million floating-point operations per second] machine, then the $(m^* + 1)(127 \text{ M/D} + 93 \text{ A/S})$ operations needed take some $220(m^* + 1)$ μsecs. to perform. This means that we have available time for some $\frac{3333}{220} - 1 \approx 14$ iterations. This should be ample.

APPENDIX

A1. THREE-DIMENSIONAL VECTOR ALGEBRA

We denote (*column*) *vectors* by italic boldface characters (e.g., $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \ldots, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$). We suppose that there is a "world coordinate frame" of reference, given by the origin O and a right-handed orthonormal triad $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. Relative to this, we write, for example,

$$\left.\begin{aligned} \boldsymbol{x} &= \xi_1 \boldsymbol{i} + \xi_2 \boldsymbol{j} + \xi_3 \boldsymbol{k} \\ \boldsymbol{y} &= \eta_1 \boldsymbol{i} + \eta_2 \boldsymbol{j} + \eta_3 \boldsymbol{k} \\ \boldsymbol{z} &= \zeta_1 \boldsymbol{i} + \zeta_2 \boldsymbol{j} + \zeta_3 \boldsymbol{k} \end{aligned}\right\}, \tag{A1}$$

or

$$\boldsymbol{x} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}, \quad \boldsymbol{z} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}. \tag{A2}$$

The *transpose* of any matrix is denoted by appending the superscript suffix $^{\mathrm{T}}$. For instance, if $\boldsymbol{x}$ is a column-vector, its row-vector transpose is $\boldsymbol{x}^{\mathrm{T}}$; thus, e.g.,

$$\boldsymbol{x}^{\mathrm{T}} = [\, \xi_1 \quad \xi_2 \quad \xi_3 \,]. \tag{A3}$$

The *scalar product* of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined to be

$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}} \boldsymbol{y} = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 \,. \tag{A4}$$

The *magnitude* ("length") of a vector $\boldsymbol{z}$ is denoted by $|\boldsymbol{z}|$ and defined to be

$$|\boldsymbol{z}|^2 = \boldsymbol{z}^2 = \boldsymbol{z}\,.\,\boldsymbol{z} = \zeta_1{}^2 + \zeta_2{}^2 + \zeta_3{}^2. \tag{A5}$$

We note that, since, by (A1),

$$\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \tag{A6}$$

it follows that

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = 1, \quad \boldsymbol{i}\,.\,\boldsymbol{j} = \boldsymbol{j}\,.\,\boldsymbol{k} = \boldsymbol{k}\,.\,\boldsymbol{i} = 0. \tag{A7}$$

It also follows from (A4) that $|\boldsymbol{z}| = 0$ if and only if $\zeta_1 = \zeta_2 = \zeta_3 = 0$. By (A4), the scalar product is clearly *linear* in both its factors.

The *angle* between vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is denoted by $\theta_{\boldsymbol{xy}}$ and is defined by the relation

$$\boldsymbol{x}\,.\,\boldsymbol{y} = \boldsymbol{y}\,.\,\boldsymbol{x} = |\boldsymbol{x}|\,|\boldsymbol{y}|\cos\theta_{\boldsymbol{xy}}. \tag{A8}$$

If $|\boldsymbol{x}| \neq 0$, $|\boldsymbol{y}| \neq 0$, and $\boldsymbol{x}\,.\,\boldsymbol{y} = 0$, we say that the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are *orthogonal*; and, by (A8), $\theta_{\boldsymbol{xy}} = \frac{\pi}{2}$; i.e., the vectors are mutually *perpendicular*. Note that $\boldsymbol{x}\,.\,\boldsymbol{y}$ is the product of the magnitude of $\boldsymbol{y}$ and the length of the *projection* of $\boldsymbol{x}$ onto $\boldsymbol{y}$ (or, of course, *vice versa*).

The *vector product* of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined to be:

$$\boldsymbol{x} \wedge \boldsymbol{y} = \begin{bmatrix} \xi_2\eta_3 - \xi_3\eta_2 \\ \xi_3\eta_1 - \xi_1\eta_3 \\ \xi_1\eta_2 - \xi_2\eta_1 \end{bmatrix}. \tag{A9}$$

It follows immediately that

$$\boldsymbol{x} \wedge \boldsymbol{y} = -\boldsymbol{y} \wedge \boldsymbol{x}. \tag{A10}$$

and that
$$\boldsymbol{z} \wedge \boldsymbol{z} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{A11}$$

Note that we may also write (A9) as a formal *determinant*:

$$\boldsymbol{x} \wedge \boldsymbol{y} = \begin{vmatrix} \boldsymbol{i} & \xi_1 & \eta_1 \\ \boldsymbol{j} & \xi_2 & \eta_2 \\ \boldsymbol{k} & \xi_3 & \eta_3 \end{vmatrix}. \tag{A12}$$

Further, by (A6) and (A9),

$$\left.\begin{aligned} \boldsymbol{i} \wedge \boldsymbol{i} = \boldsymbol{j} \wedge \boldsymbol{j} = \boldsymbol{k} \wedge \boldsymbol{k} = \mathbf{0} \\ \boldsymbol{i} \wedge \boldsymbol{j} = -\boldsymbol{j} \wedge \boldsymbol{i} = \boldsymbol{k} \\ \boldsymbol{j} \wedge \boldsymbol{k} = -\boldsymbol{k} \wedge \boldsymbol{j} = \boldsymbol{i} \\ \boldsymbol{k} \wedge \boldsymbol{i} = -\boldsymbol{i} \wedge \boldsymbol{k} = \boldsymbol{j} \end{aligned}\right\}. \tag{A13}$$

Fact 1.
$$\boldsymbol{x} \wedge \boldsymbol{y} = \Big(|\boldsymbol{x}|\,|\boldsymbol{y}| \sin \theta_{\boldsymbol{xy}}\Big)\, \boldsymbol{k}_{\boldsymbol{xy}}, \tag{A14}$$

where $\boldsymbol{k}_{\boldsymbol{xy}}$ denotes a unit vector (i.e., $|\boldsymbol{k}_{\boldsymbol{xy}}| = 1$) perpendicular to the plane of $\boldsymbol{x}$ and $\boldsymbol{y}$, so directed that $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{k}_{\boldsymbol{xy}}\}$ form a *right-handed triad.*

Proof. By (A4) and (A9),

$$\boldsymbol{x} \,.\, (\boldsymbol{x} \wedge \boldsymbol{y}) = \xi_1(\xi_2\eta_3 - \xi_3\eta_2) + \xi_2(\xi_3\eta_1 - \xi_1\eta_3) + \xi_3(\xi_1\eta_2 - \xi_2\eta_1) = 0,$$

and, similarly, $\boldsymbol{x} \,.\, (\boldsymbol{x} \wedge \boldsymbol{y}) = 0$. Thus $\boldsymbol{x} \wedge \boldsymbol{y}$ is perpendicular to the plane of $\boldsymbol{x}$ and $\boldsymbol{y}$. Also, by (A4), (A5), (A8), and (A9),

$$\begin{aligned} |\boldsymbol{x} \wedge \boldsymbol{y}|^2 &= (\xi_2\eta_3 - \xi_3\eta_2)^2 + (\xi_3\eta_1 - \xi_1\eta_3)^2 + (\xi_1\eta_2 - \xi_2\eta_1)^2 \\ &= \xi_2{}^2\eta_3{}^2 - 2\xi_2\xi_3\eta_2\eta_3 + \xi_3{}^2\eta_2{}^2 + \xi_3{}^2\eta_1{}^2 - 2\xi_1\xi_3\eta_1\eta_3 \\ &\qquad + \xi_1{}^2\eta_3{}^2 + \xi_1{}^2\eta_2{}^2 - 2\xi_1\xi_2\eta_1\eta_2 + \xi_2{}^2\eta_1{}^2 \end{aligned}$$

(CONTINUED...)

$$= (\xi_1^2 + \xi_2^2 + \xi_3^2)(\eta_1^2 + \eta_2^2 + \eta_3^2) - (\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3)^2$$

$$= |\boldsymbol{x}|^2|\boldsymbol{y}|^2 - |\boldsymbol{x}|^2|\boldsymbol{y}|^2\cos^2\theta_{\boldsymbol{xy}} = |\boldsymbol{x}|^2|\boldsymbol{y}|^2\sin^2\theta_{\boldsymbol{xy}}.$$

This proves (A14) to within a factor of ±1. Finally, consider the case of $\boldsymbol{x} = \boldsymbol{i}$ and $\boldsymbol{y} = \boldsymbol{j}$, when $\boldsymbol{x} \wedge \boldsymbol{y} = \boldsymbol{i} \wedge \boldsymbol{j} = \boldsymbol{k}$, by (A13). Since $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is a right-handed triad, $\boldsymbol{k} = \boldsymbol{k_{ij}}$. The relationship (A14) holds for all $\boldsymbol{x}$ and $\boldsymbol{y}$, by continuity considerations.

Note that the expression on the right of (A14) shows that the magnitude of $\boldsymbol{x} \wedge \boldsymbol{y}$ equals the *area* of the parallelogram with the two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ as adjacent sides.

We now consider two products of three vectors. The *triple scalar product* is defined to be $\boldsymbol{x} \,.\, (\boldsymbol{y} \wedge \boldsymbol{z})$. It follows, from (A4), (A9), and (A12), that

$$\boldsymbol{x} \,.\, (\boldsymbol{y} \wedge \boldsymbol{z}) = (\boldsymbol{x} \wedge \boldsymbol{y}) \,.\, \boldsymbol{z} = \begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{vmatrix} = |\,\boldsymbol{x}\;\;\boldsymbol{y}\;\;\boldsymbol{z}\,|. \quad \text{(A15)}$$

Fact 2. $|\,\boldsymbol{x}\;\;\boldsymbol{y}\;\;\boldsymbol{z}\,|$ equals the *volume* of the parallelepiped with $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ as adjacent sides (with the usual "right-hand rule" convention for its sign). Thus, $|\,\boldsymbol{x}\;\;\boldsymbol{y}\;\;\boldsymbol{z}\,| = 0$ if any two of the vectors are equal.

Proof. We have seen above that $\boldsymbol{t} = |\boldsymbol{y} \wedge \boldsymbol{z}| = |\boldsymbol{y}|\,|\boldsymbol{z}|\sin\theta_{\boldsymbol{yz}}$ is the *area* of the parallelogram with $\boldsymbol{y}$ and $\boldsymbol{z}$ as adjacent sides; and the vector $\boldsymbol{y} \wedge \boldsymbol{z}$ is perpendicular to the plane of this parallelogram. Furthermore, as we have also seen earlier, $\boldsymbol{x} \,.\, \boldsymbol{t} = |\boldsymbol{x}|\,|\boldsymbol{t}|\cos\theta_{\boldsymbol{xt}}$, which is the magnitude of $\boldsymbol{t}$ times the projection of $\boldsymbol{x}$ perpendicular to the plane of $\boldsymbol{y}$ and $\boldsymbol{z}$. Therefore, $\boldsymbol{x} \,.\, (\boldsymbol{y} \wedge \boldsymbol{z})$ equals the volume of the parallelepiped defined by the three vectors. By (A15) or the result just proven, if any two of the vectors are equal, the parallelepiped collapses to zero volume.

A corollary of Fact 2 is:

Fact 3.

$$+|\,\boldsymbol{x}\;\;\boldsymbol{y}\;\;\boldsymbol{z}\,| = +|\,\boldsymbol{y}\;\;\boldsymbol{z}\;\;\boldsymbol{x}\,| = +|\,\boldsymbol{z}\;\;\boldsymbol{x}\;\;\boldsymbol{y}\,|$$

$$= -|\,\boldsymbol{z}\;\;\boldsymbol{y}\;\;\boldsymbol{x}\,| = -|\,\boldsymbol{y}\;\;\boldsymbol{x}\;\;\boldsymbol{z}\,| = -|\,\boldsymbol{x}\;\;\boldsymbol{z}\;\;\boldsymbol{y}\,|. \quad \text{(A16)}$$

Fact 4. If $|\, \boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{z} \,| \neq 0$, then, for any vector $\boldsymbol{t}$,

$$\boldsymbol{t} = \frac{|\, \boldsymbol{t} \ \boldsymbol{y} \ \boldsymbol{z} \,|}{|\, \boldsymbol{x} \ \boldsymbol{y} \ \boldsymbol{z} \,|}\boldsymbol{x} + \frac{|\, \boldsymbol{t} \ \boldsymbol{z} \ \boldsymbol{x} \,|}{|\, \boldsymbol{x} \ \boldsymbol{y} \ \boldsymbol{z} \,|}\boldsymbol{y} + \frac{|\, \boldsymbol{t} \ \boldsymbol{x} \ \boldsymbol{y} \,|}{|\, \boldsymbol{x} \ \boldsymbol{y} \ \boldsymbol{z} \,|}\boldsymbol{z}. \tag{A17}$$

Proof. Since $|\, \boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{z} \,| \neq 0$, the vectors $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ form a base; so, certainly, there are unique K, L, and M, such that

$$K\boldsymbol{x} + L\boldsymbol{y} + M\boldsymbol{z} = \boldsymbol{t}. \tag{A18}$$

Thus, $|\, \boldsymbol{t} \quad \boldsymbol{y} \quad \boldsymbol{z} \,| = \boldsymbol{t} \,.\, (\boldsymbol{y} \wedge \boldsymbol{z}) = K \,|\, \boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{z} \,|$, since $|\, \boldsymbol{y} \quad \boldsymbol{y} \quad \boldsymbol{z} \,| = |\, \boldsymbol{z} \quad \boldsymbol{y} \quad \boldsymbol{z} \,| = 0$, by Fact 2. The rest of (A17) follows similarly.

Using the determinant form of (A15), we see that (A17) is nothing else than the famous *Cramér Rule* for the equations (A18).

We now turn to the *triple vector product*, $\boldsymbol{x} \wedge (\boldsymbol{y} \wedge \boldsymbol{z})$.

Fact 5.
$$\boldsymbol{x} \wedge (\boldsymbol{y} \wedge \boldsymbol{z}) = (\boldsymbol{z} \wedge \boldsymbol{y}) \wedge \boldsymbol{x} = (\boldsymbol{x} \,.\, \boldsymbol{z})\, \boldsymbol{y} - (\boldsymbol{x} \,.\, \boldsymbol{y})\, \boldsymbol{z}. \tag{A19}$$

Proof. By (A9),

$$\boldsymbol{x} \wedge (\boldsymbol{y} \wedge \boldsymbol{z}) = \begin{bmatrix} \xi_2(\eta_1\zeta_2 - \eta_2\zeta_1) - \xi_3(\eta_3\zeta_1 - \eta_1\zeta_3) \\ \xi_3(\eta_2\zeta_3 - \eta_3\zeta_2) - \xi_1(\eta_1\zeta_2 - \eta_2\zeta_1) \\ \xi_1(\eta_3\zeta_1 - \eta_1\zeta_3) - \xi_2(\eta_2\zeta_3 - \eta_3\zeta_2) \end{bmatrix}$$

$$= \begin{bmatrix} \eta_1(\xi_1\zeta_1 + \xi_2\zeta_2 + \xi_3\zeta_3) - \zeta_1(\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3) \\ \eta_2(\xi_1\zeta_1 + \xi_2\zeta_2 + \xi_3\zeta_3) - \zeta_2(\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3) \\ \eta_3(\xi_1\zeta_1 + \xi_2\zeta_2 + \xi_3\zeta_3) - \zeta_3(\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3) \end{bmatrix};$$

which proves (A19). [Of course, since $\boldsymbol{y} \wedge \boldsymbol{z}$ is clearly perpendicular to the plane of $\boldsymbol{y}$ and $\boldsymbol{z}$, it follows that $\boldsymbol{x} \wedge (\boldsymbol{y} \wedge \boldsymbol{z})$ is in the plane of $\boldsymbol{y}$ and $\boldsymbol{z}$, and perpendicular to $\boldsymbol{x}$. Hence, we get (A19) at once, to within a scalar factor; however, this factor is rather hard to determine.]

Fact 6. If $|\, \boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{z} \,| \neq 0$, then, for any vector $\boldsymbol{t}$,

$$\boldsymbol{t} = \frac{(\boldsymbol{t} \,.\, \boldsymbol{x})}{|\, \boldsymbol{x} \ \boldsymbol{y} \ \boldsymbol{z} \,|}(\boldsymbol{y} \wedge \boldsymbol{z}) + \frac{(\boldsymbol{t} \,.\, \boldsymbol{y})}{|\, \boldsymbol{x} \ \boldsymbol{y} \ \boldsymbol{z} \,|}(\boldsymbol{z} \wedge \boldsymbol{x}) + \frac{(\boldsymbol{t} \,.\, \boldsymbol{z})}{|\, \boldsymbol{x} \ \boldsymbol{y} \ \boldsymbol{z} \,|}(\boldsymbol{x} \wedge \boldsymbol{y}). \tag{A20}$$

Proof. Again, it is easy to see that the triad of vectors $\{\boldsymbol{y} \wedge \boldsymbol{z}, \boldsymbol{z} \wedge \boldsymbol{x}, \boldsymbol{x} \wedge \boldsymbol{y}\}$ form a base; so that there are unique P, Q, and R, such that $\boldsymbol{t} = P\,(\boldsymbol{y} \wedge \boldsymbol{z}) + Q\,(\boldsymbol{z} \wedge \boldsymbol{x}) + R\,(\boldsymbol{x} \wedge \boldsymbol{y})$. Thus, $\boldsymbol{t} \,.\, \boldsymbol{x} = P \,|\, \boldsymbol{x} \;\; \boldsymbol{y} \;\; \boldsymbol{z} \,|$, as before; and the rest of (A20) follows similarly.

Fact 7.

$$(\boldsymbol{x} \wedge \boldsymbol{y}) \,.\, (\boldsymbol{z} \wedge \boldsymbol{t}) = (\boldsymbol{x} \,.\, \boldsymbol{z})\,(\boldsymbol{y} \,.\, \boldsymbol{t}) - (\boldsymbol{x} \,.\, \boldsymbol{t})\,(\boldsymbol{y} \,.\, \boldsymbol{z}). \qquad \text{(A21)}$$

Proof. By (A15), with Facts 3 and 4, $(\boldsymbol{x} \wedge \boldsymbol{y}) \,.\, (\boldsymbol{z} \wedge \boldsymbol{t}) = \,|\, (\boldsymbol{x} \wedge \boldsymbol{y}) \;\; \boldsymbol{z} \;\; \boldsymbol{t} \,|$ $= \,|\, \boldsymbol{t} \;\; (\boldsymbol{x} \wedge \boldsymbol{y}) \;\; \boldsymbol{z} \,| = \boldsymbol{t} \,.\, [(\boldsymbol{x} \wedge \boldsymbol{y}) \wedge \boldsymbol{z}] = (\boldsymbol{t} \,.\, \boldsymbol{y})\,(\boldsymbol{x} \,.\, \boldsymbol{z}) - (\boldsymbol{t} \,.\, \boldsymbol{x})\,(\boldsymbol{y} \,.\, \boldsymbol{z})$; hence (A21). Similarly, by Fact 5, we get (recovering Fact 4) that

Fact 8.

$$\begin{aligned}(\boldsymbol{x} \wedge \boldsymbol{y}) \wedge (\boldsymbol{z} \wedge \boldsymbol{t}) &= |\, \boldsymbol{t} \;\; \boldsymbol{x} \;\; \boldsymbol{y} \,|\, \boldsymbol{z} - |\, \boldsymbol{x} \;\; \boldsymbol{y} \;\; \boldsymbol{z} \,|\, \boldsymbol{t} \\ &= |\, \boldsymbol{t} \;\; \boldsymbol{x} \;\; \boldsymbol{z} \,|\, \boldsymbol{y} - |\, \boldsymbol{t} \;\; \boldsymbol{y} \;\; \boldsymbol{z} \,|\, \boldsymbol{x}. \qquad \text{(A22)}\end{aligned}$$

A2. RECIPROCAL OF A (3×3) MATRIX

The determinant is defined by

$$\begin{aligned}|\boldsymbol{K}| &= \begin{vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{vmatrix} \\ &= k_{11}k_{22}k_{33} - k_{11}k_{23}k_{32} + k_{12}k_{23}k_{31} \\ &\quad - k_{12}k_{21}k_{33} + k_{13}k_{21}k_{32} - k_{13}k_{22}k_{31}. \qquad \text{(A23)}\end{aligned}$$

If we put

$$\boldsymbol{G} = \begin{bmatrix} k_{22}k_{33} - k_{23}k_{32} & k_{32}k_{13} - k_{33}k_{12} & k_{12}k_{23} - k_{13}k_{22} \\ k_{23}k_{31} - k_{21}k_{33} & k_{33}k_{11} - k_{31}k_{13} & k_{13}k_{21} - k_{11}k_{23} \\ k_{21}k_{32} - k_{22}k_{31} & k_{31}k_{12} - k_{32}k_{11} & k_{11}k_{22} - k_{12}k_{21} \end{bmatrix}, \qquad \text{(A24)}$$

then it is easily verified, by (A23), that

$$KG = GK = |K|I, \tag{A25}$$

where I denotes the *unit matrix*. Hence, if we define

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} = |\mathbf{K}|^{-1}\mathbf{G}, \tag{A26}$$

then we get that

$$H = K^{-1}. \tag{A27}$$

A3. ALGEBRAIC OPERATION COUNTS

Turning to the matter of **operation counts**, we first seek the number, $\mathcal{D}$, of FLOPs needed to evaluate a (3 × 3) *determinant*. Two methods are available to us. (a) We can use the formula in (A23), which evidently needs 6 × 2 M/D (multiplications and/or divisions—in this case, multiplications) and 5 A/S (additions and/or subtractions). Thus,

$$\mathcal{D}_{(a)} = 12\ \mathrm{M/D} + 5\ \mathrm{A/S}. \tag{A28}$$

(b) We can perform the Gaussian-elimination kind of transformations to reduce the determinant to diagonal form.

$$\begin{vmatrix} \alpha & \circledast & \circledast \\ \circledast & \circledast & \circledast \\ \circledast & \circledast & \circledast \end{vmatrix}^{[1]} \rightarrow \begin{vmatrix} 1 & \circledast & \circledast \\ \circledast & \circledast & \circledast \\ \circledast & \circledast & \circledast \end{vmatrix}^{[2]} \rightarrow \begin{vmatrix} 1 & \circledast & \circledast \\ 0 & \beta & \circledast \\ 0 & \circledast & \circledast \end{vmatrix}^{[3]}$$

$$\rightarrow \begin{vmatrix} 1 & \circledast & \circledast \\ 0 & 1 & \circledast \\ 0 & \circledast & \circledast \end{vmatrix}^{[4]} \rightarrow \begin{vmatrix} 1 & \circledast & \circledast \\ 0 & 1 & \circledast \\ 0 & 0 & \gamma \end{vmatrix}^{[5]}. \tag{A29}$$

NOTES: $\circledast$ denotes an arbitrary entry in the *tableau*.

[1] By interchanging a pair of rows, if necessary—noting that any such interchange entails a change of sign in the determinant—try to make

the (1, 1) entry, α, non-zero. If this is impossible, the first column is null and therefore the determinant must be zero; terminating the process immediately.

[2] 2 divisions by α are required in the first row. The value α is noted.

[3] 2 × (2 multiplications and 2 A/S) are required to reduce the (2, 1) and (3, 1) entries to zero. By interchanging last two rows, if necessary, try to make the (2, 2) entry, β, non-zero. If this is impossible, the determinant must be zero, and the process terminates immediately.

[4] 1 division by β is required in the second row. The value β is noted.

[5] 1 multiplication and 1 A/S are required to reduce the (3, 2) entry to zero.

The value of the determinant is $\alpha\beta\gamma$. Unless $\gamma = 0$, we need 2 multiplications to obtain this value. Combining these multiplications with the FLOPs listed in the notes above, we get

$$\mathcal{D}_{(b)} = 10 \text{ M/D} + 5 \text{ A/S}. \tag{A30}$$

(c) We can use the *triple scalar product* [see (A15)]. The generation of the *vector product* of two vectors takes 3 × (2 multiplications and 1 subtraction), yielding

$$\mathcal{V} = 6 \text{ M/D} + 3 \text{ A/S}. \tag{A31}$$

Then, the computation of the triple scalar product, by way of the scalar product, takes another 3 multiplications and 2 additions, yielding

$$\mathcal{D}_{(c)} = 9 \text{ M/D} + 5 \text{ A/S}. \tag{A32}$$

Thus, method (c) is slightly preferable and should be adopted, yielding

$$\mathcal{D} = 9 \text{ M/D} + 5 \text{ A/S}. \tag{A33}$$

Next, we seek the number, $\mathcal{R}$, of FLOPs needed to find the *reciprocal of a* (3 × 3) *matrix*. Again, there are two likely methods. (a) We can use (A24) and (A26), for which we require 9 × (2 multiplications, 1 subtraction, and 1 division); so that

$$\mathcal{R}_{(a)} = \mathcal{D} + 27 \text{ M/D} + 9 \text{ A/S}. \tag{A34}$$

(b) We can use triple Gaussian elimination. A *tableau* is set up, with the given matrix on the left and a unit matrix on the right. It is then transformed as follows.

$$
\left[\begin{array}{ccc|ccc}
\alpha & \circledast & \circledast & 1 & 0 & 0 \\
\circledast & \circledast & \circledast & 0 & 1 & 0 \\
\circledast & \circledast & \circledast & 0 & 0 & 1
\end{array}\right]^{[1]}
\rightarrow
\left[\begin{array}{ccc|ccc}
1 & \circledast & \circledast & \circledast & 0 & 0 \\
\circledast & \circledast & \circledast & 0 & 1 & 0 \\
\circledast & \circledast & \circledast & 0 & 0 & 1
\end{array}\right]^{[2]}
$$

$$
\rightarrow
\left[\begin{array}{ccc|ccc}
1 & \circledast & \circledast & \circledast & 0 & 0 \\
0 & \beta & \circledast & \circledast & 1 & 0 \\
0 & \circledast & \circledast & \circledast & 0 & 1
\end{array}\right]^{[3]}
\rightarrow
\left[\begin{array}{ccc|ccc}
1 & \circledast & \circledast & \circledast & 0 & 0 \\
0 & 1 & \circledast & \circledast & \circledast & 0 \\
0 & \circledast & \circledast & \circledast & 0 & 1
\end{array}\right]^{[4]}
$$

$$
\rightarrow
\left[\begin{array}{ccc|ccc}
1 & \circledast & \circledast & \circledast & 0 & 0 \\
0 & 1 & \circledast & \circledast & \circledast & 0 \\
0 & 0 & \gamma & \circledast & \circledast & 1
\end{array}\right]^{[5]}
\rightarrow
\left[\begin{array}{ccc|ccc}
1 & \circledast & \circledast & \circledast & 0 & 0 \\
0 & 1 & \circledast & \circledast & \circledast & 0 \\
0 & 0 & 1 & \circledast & \circledast & \circledast
\end{array}\right]^{[6]}
$$

$$
\rightarrow
\left[\begin{array}{ccc|ccc}
1 & \circledast & 0 & \circledast & \circledast & \circledast \\
0 & 1 & 0 & \circledast & \circledast & \circledast \\
0 & 0 & 1 & \circledast & \circledast & \circledast
\end{array}\right]^{[7]}
\rightarrow
\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & \circledast & \circledast & \circledast \\
0 & 1 & 0 & \circledast & \circledast & \circledast \\
0 & 0 & 1 & \circledast & \circledast & \circledast
\end{array}\right]^{[8]} .
\qquad \text{(A35)}
$$

NOTES: $\circledast$ again denotes an arbitrary entry in the *tableau*.

[1] By interchanging a pair of equations (i.e., rows), if necessary, try to make the (1, 1) entry, α, non-zero. If this is impossible, the first column is null and therefore the given matrix has no reciprocal.

[2] 3 divisions by α are required in the first rowof the *tableau*. The '1' in position (1, 4) becomes α^{-1}, a case of $\circledast$.

[3] $2 \times$ (3 multiplications and 3 A/S) are required to reduce the (2, 1) and (3, 1) entries to zero. By interchanging last two equations, if necessary, try to make the (2, 2) entry, β, non-zero. If this is impossible, again the given matrix has no reciprocal.

[4] 3 divisions by β are required in the second rowof the *tableau*. The '1' in position (2, 5) becomes β^{-1}, a case of $\circledast$.

[5] 3 multiplications and 3 A/S are required to reduce the (3, 2) entry to zero. Hopefully, the (3, 3) entry, γ, is not zero. If $\gamma = 0$, then the given matrix has no reciprocal.

[6] 3 divisions by γ are required in the third row of the *tableau*. The '1' in position (3, 6) becomes γ^{-1}, a case of $\circledast$.

[7] Begin the "back-substitution" process. $2 \times$ (3 multiplications and 3 A/S) are required to reduce the (2, 3) and (1, 3) entries to zero. The right half of the *tableau* fills with $\circledast$.

[8] 3 multiplications and 3 A/S are required to reduce the (1, 2) entry to zero.

The left half of the *tableau* is transformed into a unit matrix, and the right half becomes the required reciprocal matrix. Combining the FLOPs listed in the notes above, we get

$$\mathcal{R}_{(b)} = 27 \text{ M/D} + 18 \text{ A/S}. \tag{A36}$$

If we require the value of the determinant of the matrix, too, it is no extra work to note the coefficients α, β, and γ, as we go along in (A33), and the determinant value, $\alpha\beta\gamma$, is obtained by only two additional multiplications. Clearly, the second method is again preferable, yielding

$$\mathcal{R} = 27 \text{ M/D} + 18 \text{ A/S}. \tag{A37}$$

and

$$\mathcal{D}_{\text{with } \mathcal{R}} = 2 \text{ M/D}. \tag{A38}$$

Very analogous is the problem of the number $\mathcal{G}$ of FLOPs required to find a single vector solution of a system of three equations in three unknowns. Again, Gaussian elimination is a prime candidate, and the *tableau* sequence is very similar to (A35) above.

$$\left[\begin{array}{ccc|c} \alpha & \circledast & \circledast & \circledast \\ \circledast & \circledast & \circledast & \circledast \\ \circledast & \circledast & \circledast & \circledast \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & \circledast & \circledast & \circledast \\ \circledast & \circledast & \circledast & \circledast \\ \circledast & \circledast & \circledast & \circledast \end{array}\right]^{[1]} \rightarrow \left[\begin{array}{ccc|c} 1 & \circledast & \circledast & \circledast \\ 0 & \beta & \circledast & \circledast \\ 0 & \circledast & \circledast & \circledast \end{array}\right]^{[2]}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & \circledast & \circledast & \circledast \\ 0 & 1 & \circledast & \circledast \\ 0 & \circledast & \circledast & \circledast \end{array}\right]^{[3]} \rightarrow \left[\begin{array}{ccc|c} 1 & \circledast & \circledast & \circledast \\ 0 & 1 & \circledast & \circledast \\ 0 & 0 & \gamma & \circledast \end{array}\right]^{[4]} \rightarrow \left[\begin{array}{ccc|c} 1 & \circledast & \circledast & \circledast \\ 0 & 1 & \circledast & \circledast \\ 0 & 0 & 1 & \circledast \end{array}\right]^{[5]}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & \circledast & 0 & \circledast \\ 0 & 1 & 0 & \circledast \\ 0 & 0 & 1 & \circledast \end{array}\right]^{[6]} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \circledast \\ 0 & 1 & 0 & \circledast \\ 0 & 0 & 1 & \circledast \end{array}\right]^{[7]} . \tag{A39}$$

NOTES:

[1] 3 divisions by α are required in the first rowof the *tableau.*

[2] 2 × (3 multiplications and 3 A/S) are required to reduce the (2, 1) and (3, 1) entries to zero.

[3] 2 divisions by β are required in the second rowof the *tableau.*

[4] 2 multiplications and 2 A/S are required to reduce the (3, 2) entry to zero.

[5] 1 division by γ is required in the third rowof the *tableau.*

[6] Begin the "back-substitution" process. 2 × (1 multiplication and 1 A/S) are required to reduce the (2, 3) and (1, 3) entries to zero.

[7] 1 multiplication and 1 A/S is required to reduce the (1, 2) entry to zero.

We get

$$\mathcal{G} = 17\ \text{M/D} + 11\ \text{A/S}. \tag{A40}$$

It is easily verified that the *multiplication of two* (3 × 3) *matrices* will require 9 × (3 multiplications and 2 additions), yielding

$$\mathcal{M} = 27\ \text{M/D} + 18\ \text{A/S}. \tag{A41}$$

A4. MULTI-DIMENSIONAL NEWTON METHOD

We begin with the one-dimensional Newton method for iteratively computing the solution ξ of the equation

$$f(\xi) = 0. \tag{A42}$$

Taylor's expansion is

$$f(x + h) = f(x) + h\frac{\partial f(x)}{\partial x} + \frac{1}{2}h^2\frac{\partial^2 f(x)}{\partial x^2} + \ldots \tag{A43}$$

Newton's method (sometimes referred to as the *Newton-Raphson method*) consists of selecting a suitable initial iterate (guess), $x^{(0)}$, approximating ξ; and, thereafter, using the *linearized* form of (A43) to obtain the $(m+1)$-st iterate, $x^{(m+1)}$, from the m-th iterate, $x^{(m)}$:

$$f(x^{(m+1)}) \approx 0 = f(x^{(m)}) + (x^{(m+1)} - x^{(m)})\frac{\partial f(x^{(m)})}{\partial x^{(m)}}. \tag{A44}$$

This is usually written in the form

$$x^{(m+1)} = x^{(m)} - \frac{f(x^{(m)})}{f'(x^{(m)})}. \tag{A45}$$

If we write $$\varepsilon^{(m)} = x^{(m)} - \xi \tag{A46}$$

for the *error* in the *m*-th iterate, so that

$$x^{(m)} = \xi + \varepsilon^{(m)} \quad \text{and} \quad x^{(m+1)} = \xi + \varepsilon^{(m+1)}, \tag{A47}$$

then (A42) – (A44) yield that

$$\begin{aligned} 0 &= f(\xi) + \varepsilon^{(m)} \frac{\partial f(\xi)}{\partial \xi} + \frac{1}{2} \varepsilon^{(m)2} \frac{\partial^2 f(\xi)}{\partial \xi^2} + \dots \\ &\qquad + [\varepsilon^{(m+1)} - \varepsilon^{(m)}] \left\{ \frac{\partial f(\xi)}{\partial \xi} + \varepsilon^{(m)} \frac{\partial^2 f(\xi)}{\partial \xi^2} + \dots \right\} \\ &= \varepsilon^{(m+1)} \frac{\partial f(\xi)}{\partial \xi} \left\{ 1 + O[\varepsilon^{(m)}] \right\} - \frac{1}{2} \varepsilon^{(m)2} \frac{\partial^2 f(\xi)}{\partial \xi^2} \left\{ 1 + O[\varepsilon^{(m)}] \right\}; \end{aligned}$$

so that, if $\frac{\partial f(\xi)}{\partial \xi}$ and $\frac{\partial^2 f(\xi)}{\partial \xi^2}$ are non-zero (which is true in most cases), then

$$\varepsilon^{(m+1)} \frac{\partial f(\xi)}{\partial \xi} \sim \frac{1}{2} \varepsilon^{(m)2} \frac{\partial^2 f(\xi)}{\partial \xi^2} \quad \text{as} \quad \varepsilon^{(m)} \to 0. \tag{A48}$$

This is referred to as *quadratic convergence.*

If we now consider a *system of n equations in n unknowns,*

$$f_i(\xi_1, \dots, \xi_n) = 0, \quad i = 1, \dots, n; \tag{A49}$$

then *Taylor's expansion* becomes (for $i = 1, \dots, n$)

$$f_i(\boldsymbol{x} + \boldsymbol{h}) = f_i(x_1 + h_1, \ldots, x_n + h_n)$$

$$= f_i(\boldsymbol{x}) + \sum_{j=1}^{n} h_j \frac{\partial f_i(\boldsymbol{x})}{\partial x_j} + \frac{1}{2} \sum_{j=1}^{n} \sum_{j=1}^{n} h_j h_k \frac{\partial^2 f_i(\boldsymbol{x})}{\partial x_j \partial x_k} + \ldots, \quad \text{(A50)}$$

and *Newton's method* becomes

$$0 = f_i(\boldsymbol{x}^{(m)}) + \sum_{j=1}^{n} (x_j^{(m+1)} - x_j^{(m)}) \frac{\partial f_i(x^{(m)})}{\partial x_j^{(m)}}. \quad \text{(A51)}$$

Writing

$$\varepsilon_j^{(m)} = x_j^{(m)} - \xi_j \quad \text{(A52)}$$

for the *error* in the j-th component of the m-th iterate, we can use (A49) - (A51), much as before, to yield

$$0 = \sum_{j=1}^{n} \varepsilon_j^{(m)} \frac{\partial f_i(\xi_1, \ldots, \xi_n)}{\partial x_j} + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \varepsilon_j^{(m)} \varepsilon_k^{(m)} \frac{\partial^2 f_i(\xi_1, \ldots, \xi_n)}{\partial x_j \partial x_k} + \ldots$$

$$+ \sum_{j=1}^{n} (\varepsilon_j^{(m+1)} - \varepsilon_j^{(m)}) \left\{ \frac{\partial f_i(\xi_1, \ldots, \xi_n)}{\partial x_j} + \sum_{k=1}^{n} \varepsilon_k^{(m)} \frac{\partial^2 f_i(\xi_1, \ldots, \xi_n)}{\partial x_j \partial x_k} + \ldots \right\}$$

$$= \sum_{j=1}^{n} \varepsilon_j^{(m+1)} \frac{\partial f_i(\xi_1, \ldots, \xi_n)}{\partial x_j} \left\{ 1 + O\left[\max_j |\varepsilon_j^{(m)}| \right] \right\}$$

$$- \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \varepsilon_j^{(m)} \varepsilon_k^{(m)} \frac{\partial^2 f_i(\xi_1, \ldots, \xi_n)}{\partial x_j \partial x_k} \left\{ 1 + O\left[\max_j |\varepsilon_j^{(m)}| \right] \right\}; \quad \text{(A53)}$$

so that, if at least one each of the partial first derivatives $\frac{\partial f_i(\xi_1, \ldots, \xi_n)}{\partial x_j}$ and of the partial second derivatives $\frac{\partial^2 f_i(\xi_1, \ldots \xi_n)}{\partial x_j \partial x_k}$ are non-zero (which is again true in most cases), then

$$\sum_{j=1}^{n} \varepsilon_j^{(m+1)} \frac{\partial f_i(\xi_1, \ldots, \xi_n)}{\partial x_j} \sim \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \varepsilon_j^{(m)} \varepsilon_k^{(m)} \frac{\partial^2 f_i(\xi_1, \ldots, \xi_n)}{\partial x_j \partial x_k}$$

$$\text{as} \quad \max_j |\varepsilon_j^{(m)}| \to 0. \qquad \text{(A54)}$$

Thus the *quadratic convergence* is preserved.

It is relatively simple to verify that the relation (A54) is satisfied by the asymptotic form

$$\max_j |\varepsilon_j^{(m)}| \sim K \kappa^{(2^m)}, \qquad \text{(A55)}$$

where K is a positive constant (depending on the functions $f_1, \ldots, f_n$) and κ is another, satisfying $0 \leq \kappa < 1$.

A5. The Two-Dimensional Problem

It is tempting to consider a simplified problem, in which everything occurs in the plane (two dimensions), rather than in three-dimensional space. It is intuitively plausible to consider a two-camera system, in this case; but let us temporarily retain the three cameras. Referring to §2 and adapting to two dimensions, we find that (2.4) simplifies to only five equations; namely,

$$\left.\begin{aligned} \boldsymbol{u}^2 &= u_1^2 + u_2^2 = a \\ \boldsymbol{v}^2 &= v_1^2 + v_2^2 = b \\ \boldsymbol{w}^2 &= w_1^2 + w_2^2 = c \\ \boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v} &= v_1 w_1 + v_2 w_2 = d \\ \boldsymbol{w} \cdot \boldsymbol{u} = \boldsymbol{u} \cdot \boldsymbol{w} &= w_1 u_1 + w_2 u_2 = e \end{aligned}\right\}; \qquad \text{(A56)}$$

whose solutions satisfy, for some ψ,

$$\left.\begin{aligned} u_1 &= \frac{e\cos\psi - \sqrt{ac-e^2}\sin\psi}{\sqrt{c}}, \quad u_2 = \frac{e\sin\psi + \sqrt{ac-e^2}\cos\psi}{\sqrt{c}} \\ v_1 &= \frac{d\cos\psi - \sqrt{bc-d^2}\sin\psi}{\sqrt{c}}, \quad v_2 = \frac{d\sin\psi + \sqrt{bc-d^2}\cos\psi}{\sqrt{c}} \\ w_1 &= \sqrt{c}\cos\psi, \quad w_2 = \sqrt{c}\sin\psi \end{aligned}\right\}. \tag{A57}$$

The last equation of (2.4) becomes

$$(cf - de)^2 = (bc - d^2)(ac - e^2), \tag{A58}$$

which is a relation among a, b, c, d, e, and f—redundant, for our purposes. The equations(2.11) reduce to

$$\left.\begin{aligned} s_1 + u_1 + \lambda p_1 &= A_1 \\ s_2 + u_2 + \lambda p_2 &= A_2 \\ s_1 + v_1 + \mu q_1 &= B_1 \\ s_2 + v_2 + \mu q_2 &= B_2 \\ s_1 + w_1 + \nu r_1 &= C_1 \\ s_2 + w_2 + \nu r_2 &= C_2 \end{aligned}\right\}. \tag{A59}$$

We can eliminate λ, μ, and ν as before, leaving us with three equations for the three remaining unknowns, ψ, s_1, and s_2. Thus, a solution is likely.

By contrast, if we follow intuition and limit ourselves to only two cameras, then (A56) further shrinks to

$$\left.\begin{aligned} \boldsymbol{v}^2 &= v_1{}^2 + v_2{}^2 = b \\ \boldsymbol{w}^2 &= w_1{}^2 + w_2{}^2 = c \\ \boldsymbol{v}\,.\,\boldsymbol{w} = \boldsymbol{w}\,.\,\boldsymbol{v} &= v_1 w_1 + v_2 w_2 = d \end{aligned}\right\}; \tag{A60}$$

whose solutions satisfy, for some ψ,

$$\left.\begin{aligned} v_1 &= \frac{d\cos\psi - \sqrt{bc - d^2}\sin\psi}{\sqrt{c}}, \quad v_2 = \frac{d\sin\psi + \sqrt{bc - d^2}\cos\psi}{\sqrt{c}} \\ w_1 &= \sqrt{c}\cos\psi, \quad w_2 = \sqrt{c}\sin\psi \end{aligned}\right\} \tag{A61}$$

[compare (A57)]; and (A59) diminishes to

$$\left.\begin{aligned} s_1 + v_1 + \mu q_1 &= B_1 \\ s_2 + v_2 + \mu q_2 &= B_2 \\ s_1 + w_1 + \nu r_1 &= C_1 \\ s_2 + w_2 + \nu r_2 &= C_2 \end{aligned}\right\}. \tag{A62}$$

and, when we eliminate λ and μ, we are left with two equations for the same three unknowns, ψ, s_1, and s_2. Thus, *a solution generally cannot be uniquely determined*. Indeed, for *any* choice of the angle ψ, we can get $\boldsymbol{v}$ and $\boldsymbol{w}$ from (A61); whence $\boldsymbol{q}$ and $\boldsymbol{r}$ are determined by the reduced form of (2.5),

$$\left.\begin{aligned} \boldsymbol{q} &= k_{22}\,\boldsymbol{v} + k_{32}\,\boldsymbol{w} \\ \boldsymbol{r} &= k_{23}\,\boldsymbol{v} + k_{33}\,\boldsymbol{w} \end{aligned}\right\}; \tag{A63}$$

and then s_1 and s_2 follow from [compare (4.1)]

$$\left.\begin{aligned} s_1 &= \frac{q_2 r_1(B_1 - u_1) - q_1 r_2(C_1 - v_1) - q_1 r_1(B_2 - C_2 + v_2 - u_2)}{q_2 r_1 - q_1 r_2} \\ s_2 &= \frac{q_1 r_2(B_2 - u_2) - q_2 r_1(C_2 - v_2) - q_2 r_2(B_1 - C_1 + v_1 - u_1)}{q_1 r_2 - q_2 r_1} \end{aligned}\right\}. \tag{A64}$$

In this case, intuition turns out to be entirely misleading!

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